

Human Reliability and Error

Basic Mathematical Concepts

2.1 Introduction

The origin of the word “mathematics” may be traced back to the Greek word “mathema,” which means “science, knowledge, or learning.” However, our present number symbols first appeared on the stone columns erected by the Scythian Indian Emperor Asoka around 250 B.C. [1, 2]. Over the centuries, mathematics has branched out into many specialized areas such as pure mathematics, applied mathematics, and probability and statistics.

Needless to say, today mathematics plays an important role in finding solutions to various types of science and engineering related problems. Its application ranges from solving planetary problems to designing systems for use in the area of transportation. Over the past many decades, mathematical concepts such as probability distributions and stochastic processes (Markov modeling) have also been used to perform various types of human reliability and error analyses. For example, in the late 1960s and early 1970s various probability distributions were used to represent times to human error [3–5]. Furthermore, in the early 1980s, the Markov method was used to perform various types of human reliability-related analysis [6–8]. This chapter presents various mathematical concepts considered useful to perform human reliability and error analyses in transportation systems.

2.2 Sets, Boolean Algebra Laws, Probability Definition, and Probability Properties

Sets play an important role in probability theory. A set may simply be described as any well-defined list, collection, or class of objects. The backbone of the axiomatic probability is set theory and sets are usually called events. Usually, sets are denoted by capital letters A, B, C, Two basic set operations are as follows [9–10]:

- **Union of Sets.** The symbol + or U is used to denote union of sets. The union of sets/events, say M and N , is the set, say D , of all elements which belong to M or to N or to both. This is expressed as follows:

$$D = M + N. \tag{2.1}$$

- **Intersection of Sets.** The symbol \cap or dot (\cdot) (or no dot at all) is used to denote intersection of sets. For example, if the intersection of sets or events M and N is denoted by a third set, say L , then this set contains all elements which belong to both M and N . This is expressed as follows:

$$L = M \cap N, \tag{2.2}$$

or

$$L = M \cdot N, \tag{2.3}$$

or

$$L = MN. \tag{2.4}$$

The Venn diagram in Fig. 2.1 shows the above case. If there are no common elements between sets M and N (i.e., $M \cap N = 0$), then these two sets are called mutually exclusive or disjoint sets.

Some of the basic laws of Boolean algebra are presented in Table 2.1 [10–11]. Capital letters M , N , and Z in the table denote sets or events.

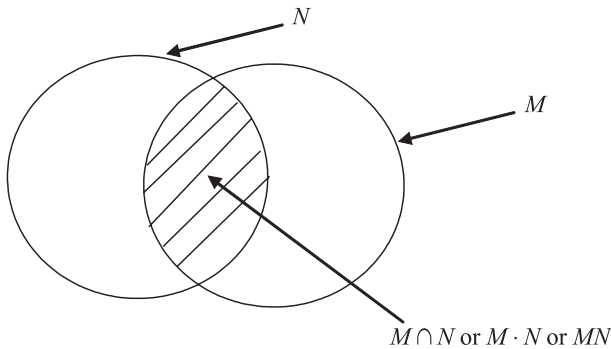


Figure 2.1. Venn diagram for the intersection of sets N and M

Table 2.1. Some basic laws of Boolean algebra

No.	Law Description	Law
1	Idempotent Laws	$M \cdot M = M$ $M + M = M$
2	Absorption Laws	$M (M \cdot N) = M \cdot N$ $M + (M \cdot N) = M$
3	Commutative Laws	$M + N = N + M$ $M \cdot N = N \cdot M$
4	Distributive Laws	$Z (M + N) = (Z \cdot M) + (Z \cdot N)$ $Z + (M \cdot N) = (Z + M) \cdot (Z + N)$

Mathematically, probability is defined as follows [12, 13] :

$$P(X) = \lim_{n \rightarrow \infty} \left(\frac{N}{n} \right), \tag{2.5}$$

where

N is the number of times event X occurs in n repeated trials or experiments.

$P(X)$ is the probability of occurrence of event X .

The basic properties of probability are as follows [9, 10–12]:

- The probability of occurrence of an event, say A , is always

$$0 \leq P(A) \leq 1. \tag{2.6}$$

- The probability of occurrence and nonoccurrence of an event A is always

$$P(A) + P(\bar{A}) = 1, \tag{2.7}$$

where

$P(A)$ is the probability of occurrence of event A .

$P(\bar{A})$ is the probability of nonoccurrence of event A .

- The probability of the sample space S is

$$P(S) = 1. \tag{2.8}$$

- The probability of negation of the sample space S is

$$P(\bar{S}) = 0. \tag{2.9}$$

- The probability of union of n independent events $X_1, X_2, X_3, \dots, X_n$ is expressed by

$$P(X_1 + X_2 + X_3 + \dots + X_n) = 1 - \prod_{i=1}^n (1 - P(X_i)), \tag{2.10}$$

where

$P(X_i)$ is the probability of occurrence of event X_i ; for $i = 1, 2, 3, \dots, n$.

- The probability of union of n mutually exclusive events $X_1, X_2, X_3, \dots, X_n$ is

$$P(X_1 + X_2 + X_3 + \dots + X_n) = \sum_{i=1}^n P(X_i). \tag{2.11}$$

- The probability of intersection of n independent events $X_1, X_2, X_3, \dots, X_n$ is

$$P(X_1 X_2 X_3 \dots X_n) = \prod_{i=1}^n P(X_i). \tag{2.12}$$

Example 2.1

Assume that a transportation system operation task is being performed by two independent individuals: A and B . The task will not be performed correctly if either of the individuals makes an error. The probabilities of making an error by individuals A and B are 0.3 and 0.2, respectively. Calculate the probability that the task will not be accomplished successfully.

Thus for $n=2$, from Equation (2.10), we get

$$P(A+B)=1-(1-P(A))(1-P(B)), \quad (2.13)$$

where

$$A=X_1 \text{ and } B=X_2.$$

By substituting the specified probability values into Equation (2.13), we get

$$\begin{aligned} P(A+B) &= 1 - (1 - 0.3)(1 - 0.2), \\ &= 0.44. \end{aligned}$$

Thus, the probability of not accomplishing the task correctly is 0.44.

2.3 Useful Mathematical Definitions

This section presents some mathematical definitions that are considered useful to perform human reliability and error analysis in transportation systems.

2.3.1 Cumulative Distribution Function Type I

For continuous random variables, this is defined by [13]

$$F(t) = \int_{-\infty}^t f(x) dx, \quad (2.14)$$

where

t is a continuous random variable (e.g., time).

$F(t)$ is the cumulative distribution function.

$f(t)$ is the probability density function.

For $t = \infty$, Equation (2.14) yields

$$\begin{aligned} F(\infty) &= \int_{-\infty}^{\infty} f(x) dx, \\ &= 1. \end{aligned}$$

This simply means that the total area under the probability density curve is always equal to unity.

2.3.2 Probability Density Function Type I

For a single-dimension discrete random variable Y , the discrete probability function of the random variable Y is represented by $f(y_i)$ if the following conditions apply:

$$f(y_i) \geq 0, \quad \text{for all } y_i \in R_y \text{ (range space),} \quad (2.15)$$

and

$$\sum_{\substack{\text{all} \\ y_i}} f(y_i) = 1. \quad (2.16)$$

2.3.3 Cumulative Distribution Function Type II

For discrete random variables, the cumulative distribution function is defined by

$$F(y) = \sum_{y_i \leq y} f(y_i), \quad (2.17)$$

where

$F(y)$ is the cumulative distribution function.

It is to be noted that the value of $F(y)$ is always

$$0 \leq F(y) \leq 1. \quad (2.18)$$

2.3.4 Probability Density Function Type II

For continuous random variables, using Equation (2.14) this is expressed as

$$\begin{aligned} \frac{dF(t)}{dt} &= \frac{d \left[\int_{-\infty}^t f(x) dx \right]}{dt}, \\ &= f(t). \end{aligned} \quad (2.19)$$

2.3.5 Expected Value Type I

The expected value, $E(t)$, of a continuous random variable is defined by [12, 13]:

$$E(t) = \mu = \int_{-\infty}^{\infty} t f(t) dt, \quad (2.20)$$

where

μ is the mean value.

t is a continuous random variable.

$f(t)$ is the probability density function.

In human reliability work, μ is known as mean time to human error, and $f(t)$ as probability density of times to human error [14].

2.3.6 Expected Value Type II

The expected value, $E(y)$, of a discrete random variable is defined by [12, 13]

$$E(y) = \sum_{i=1}^n y_i f(y_i), \tag{2.21}$$

where

n is the number of discrete values of the random variable y .

2.3.7 Laplace Transform

The Laplace transform of the function $f(t)$ is defined by

$$f(s) = \int_0^{\infty} f(t) e^{-st} dt, \tag{2.22}$$

where

t is the time variable.

s is the Laplace transform variable.

$f(s)$ is the Laplace transform of $f(t)$.

Laplace transforms of some commonly occurring functions in human reliability work are presented in Table 2.2 [15].

Table 2.2. Laplace transforms of selected functions

$f(t)$	$f(s)$
$e^{-\lambda t}$	$1/(s + \lambda)$
$t e^{-\lambda t}$	$1/(s + \lambda)^2$
$\frac{d f(t)}{dt}$	$s f(s) - f(0)$
c (a constant)	c/s
$\int_0^t f(t) dt$	$f(s)/s$
t^n , for $n = 0, 1, 2, 3, \dots$	$n! / s^{n+1}$

2.3.8 Laplace Transform: Final-value Theorem

If the following limits exist, then the final-value theorem may be expressed as follows [16]:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} [s f(s)]. \quad (2.23)$$

2.4 Solving First-order Differential Equations with Laplace Transforms

In performing human reliability and error analyses of transportation systems, solutions to first-order linear differential equations may have to be found. The use of Laplace transforms is considered to be an effective method to find solutions to such equations. The following example demonstrates the application of Laplace transforms to find solution to a system of first order differential equations.

Example 2.2

Assume that the following three first-order linear differential equations describe a fluid flow valve being in three distinct states: 0 (working normally), 1 (failed in open mode), 2 (failed in closed mode):

$$\frac{dP_0(t)}{dt} + (\lambda_0 + \lambda_c) P_0(t) = 0, \quad (2.24)$$

$$\frac{dP_1(t)}{dt} - \lambda_0 P_0(t) = 0, \quad (2.25)$$

$$\frac{dP_2(t)}{dt} - \lambda_c P_0(t) = 0. \quad (2.26)$$

At time $t=0$, $P_0(0)=1$, and $P_1(0)=P_2(0)=0$.

The symbols used in Equations (2.24)–(2.26) are defined below.

$P_i(t)$ is the probability that the fluid valve is in state i at time t , for

- $i=0$ (working normally),
- $i=1$ (failed in open mode), and
- $i=2$ (failed in closed mode).

λ_0 is the constant open mode failure rate of the fluid flow valve.

λ_c is the constant close mode failure rate of the fluid flow valve.

Find solutions to Equations (2.24)–(2.26) by using Laplace transforms.

By taking Laplace transforms of Equations (2.24)–(2.26) and using initial conditions, we get

$$s P_0(s) + (\lambda_0 + \lambda_c) P_0(s) = 1, \quad (2.27)$$

$$s P_1(s) - \lambda_0 P_0(s) = 0, \quad (2.28)$$

$$s P_2(s) - \lambda_c P_c(s) = 0. \quad (2.29)$$

By solving Equations (2.27)–(2.29), we obtain

$$P_0(s) = \frac{1}{s + \lambda_0 + \lambda_c}, \quad (2.30)$$

$$P_1(s) = \frac{\lambda_0}{s(s + \lambda_0 + \lambda_c)}, \quad (2.31)$$

$$P_2(s) = \frac{\lambda_c}{s(s + \lambda_0 + \lambda_c)}. \quad (2.32)$$

Taking inverse Laplace transforms of Equations (2.30)–(2.32) yields

$$P_0(t) = e^{-(\lambda_0 + \lambda_c)t}, \quad (2.33)$$

$$P_1(t) = \frac{\lambda_0}{(\lambda_0 + \lambda_c)} \left[1 - e^{-(\lambda_0 + \lambda_c)t} \right], \quad (2.34)$$

$$P_2(t) = \frac{\lambda_c}{(\lambda_0 + \lambda_c)} \left[1 - e^{-(\lambda_0 + \lambda_c)t} \right]. \quad (2.35)$$

Thus, Equations (2.33)–(2.35) are the solutions to differential Equations (2.24)–(2.26).

2.5 Probability Distributions

There are many discrete and continuous random variable probability distributions. This section presents some of these distributions considered useful for application in performing human reliability and error analyses in transportation systems [17].

2.5.1 Binomial Distribution

The binomial distribution is a discrete random variable distribution and is also known as the Bernoulli distribution after its originator, Jakob Bernoulli (1654–1705) [1]. The distribution becomes useful in situations where one is concerned

with the probability of outcome such as the total number of failures or errors in a sequence of, say n , trials. However, it is to be noted that the binomial distribution is based upon the reasoning that each trial has two possible outcomes (e.g., success and failure) and the probability of each trial remains constant.

The binomial probability density function, $f(x)$, is defined by

$$f(x) = \binom{n}{x} p^x q^{n-x}, \quad \text{for } x=0, 1, 2, \dots, n, \quad (2.36)$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

x is the number of failures in n trials.

p is the single trial probability of success.

q is the single trial probability of failure.

The cumulative distribution function is given by

$$F(x) = \sum_{i=0}^x \binom{n}{i} p^i q^{n-i}, \quad (2.37)$$

where

$F(x)$ is the cumulative distribution function or the probability of x or less failures in n trials.

The distribution mean is given by [17]

$$\mu_b = n p, \quad (2.38)$$

where

μ_b is the mean of the binomial distribution.

2.5.2 Poisson Distribution

This is another discrete random variable distribution, named after Simeon Poisson (1781–1840) [1]. The Poisson distribution is used in situations where one is interested in the occurrence of a number of events that are of the same type. Each event's occurrence is denoted as a point on a time scale, and in reliability work each event represents a failure (error).

The Poisson density function is defined by

$$f(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad \text{for } n=0, 1, 2, \dots, \quad (2.39)$$

where

t is time.

λ is the constant failure, arrival, or error rate.

The cumulative distribution function is given by

$$F = \sum_{i=0}^n \frac{(\lambda t)^i e^{-\lambda t}}{i!}, \quad (2.40)$$

where

F is the cumulative distribution function.

The distribution mean is given by [17]

$$\mu_p = \lambda t, \quad (2.41)$$

where

μ_p is the mean of the Poisson distribution.

2.5.3 Exponential Distribution

The exponential distribution is a continuous random variable distribution and is probably the most widely used distribution in reliability work, because it is relatively easy to handle in performing reliability analysis. Another important reason for its widespread use in the industrial sector is that many engineering items exhibit constant failure rate during their useful life [18].

The distribution probability density function is defined by

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad \lambda > 0, \quad (2.42)$$

where

$f(t)$ is the probability density function.

λ is the distribution parameter. In human reliability work, it is known as the constant error rate.

t is time.

By substituting Equation (2.42) into Equation (2.14), we get

$$F(t) = 1 - e^{-\lambda t}. \quad (2.43)$$

Using Equation (2.42) in Equation (2.20) yields

$$E(t) = \mu = \frac{1}{\lambda}. \quad (2.44)$$

When λ is expressed in the term of human errors/unit time (e.g., errors/hour), Equation (2.44) gives mean time to human error (MTTHE).

Example 2.3

Assume that the constant error rate of a transit system operator is 0.0005 errors/hour. Calculate the operator's unreliability for an 8-hour mission and mean time to human error.

By substituting the given data values into Equations (2.43) and (2.44), we get

$$F(8) = 1 - e^{-(0.0005)(8)},$$

$$= 0.004,$$

and

$$E(t) = \mu = 2,000.$$

Thus, the operator's unreliability and mean time to human error are 0.004 and 2,000 hours, respectively.

2.5.4 Rayleigh Distribution

The Rayleigh distribution is another continuous random variable distribution and is often used in reliability studies. The distribution is named after John Rayleigh (1842–1919), its originator [1]. The Rayleigh distribution can be used to predict a transit system operator's reliability when his/her error rate increases linearly with time.

The distribution probability density function is defined by

$$f(t) = \frac{2}{\beta^2} t e^{-\left(\frac{t}{\beta}\right)^2}, \quad t \geq 0, \quad \beta > 0, \quad (2.45)$$

where

β is the distribution parameter.

By inserting Equation (2.45) into Equation (2.14), we get

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^2}. \quad (2.46)$$

Substituting Equation (2.45) into Equation (2.20) yields

$$E(t) = \mu = \beta \Gamma\left(\frac{3}{2}\right), \quad (2.47)$$

where

$\Gamma(\cdot)$ is the gamma function and is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \text{for } x > 0. \quad (2.48)$$

2.5.5 Weibull Distribution

The Weibull distribution is a continuous random variable distribution that is often used in reliability work. It was developed by W. Weibull (1887–1979), a Swedish

mechanical engineering professor, in the early 1950s [19]. The probability density function of the distribution is defined by

$$f(t) = \frac{bt^{b-1}}{\beta^b} e^{-\left(\frac{t}{\beta}\right)^b}, \quad t \geq 0, b > 0, \beta > 0, \tag{2.49}$$

where

b and β are the distribution shape and scale parameters, respectively.

By inserting Equation (2.49) into Equation (2.14), we obtain the following cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^b}. \tag{2.50}$$

Using Equation (2.49) in Equation (2.20), we obtain the following equation for the expected value of t :

$$E(t) = \mu = \beta \Gamma\left(1 + \frac{1}{b}\right). \tag{2.51}$$

For $b=1$ and $b=2$, Equations (2.49)–(2.51) become equations for exponential and Rayleigh distributions, respectively. This simply means that exponential and Rayleigh distributions are the special cases of the Weibull distribution.

2.5.6 Gamma Distribution

The gamma distribution is a two-parameter distribution that is quite flexible to study a wide variety of problems including those of human reliability and errors. The distribution probability density function is defined by [16]:

$$f(t) = \frac{\lambda(\lambda t)^{b-1}}{\Gamma(b)} e^{-\lambda t}, \quad t \geq 0, b > 0, \lambda > 0, \tag{2.52}$$

where

$\Gamma(\cdot)$ is the gamma function.

b and λ are the distribution shape and scale parameters, respectively.

Using Equations (2.14) and (2.52), we get the following cumulative distribution function:

$$F(t) = 1 - \sum_{i=0}^{b-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}. \tag{2.53}$$

By substituting Equation (2.52) into Equation (2.20), we get the following expression for the expected value of t :

$$E(t) = \mu = \frac{b}{\lambda}. \tag{2.54}$$

It is to be noted that for $b=1$, the gamma distribution becomes the exponential distribution.

2.5.7 Log-normal Distribution

The log-normal distribution is another two-parameter distribution that can be used to represent times to operator errors. The distribution probability density function is defined by

$$f(t) = \frac{1}{(\sqrt{2\pi})t\alpha} \exp \left[-\frac{(\ln t - m)^2}{2\alpha^2} \right], \quad t \geq 0, \quad (2.55)$$

where

α and m are the distribution parameters.

Using Equation (2.55) in Equation (2.14) yields

$$F(t) = \frac{1}{(\sqrt{2\pi})\alpha} \int_0^t \frac{1}{x} \exp \left[-\frac{(\ln x - m)^2}{2\alpha^2} \right] dx. \quad (2.56)$$

Letting $w = \frac{(\ln t - m)}{\alpha}$, we get

$$\frac{dw}{dx} = \frac{1}{\alpha x}. \quad (2.57)$$

Therefore,

$$dw = \frac{dx}{\alpha x}. \quad (2.58)$$

Using Equations (2.57) and (2.58) in Equation (2.56), we get

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(\ln t - m)}{\alpha}} \exp \left[-\frac{w^2}{2} \right] dw. \quad (2.59)$$

2.5.8 Normal Distribution

The normal distribution is a well known distribution that is also known as the Gaussian distribution after Carl Friedrich Gauss (1777–1855), a German mathematician. The probability density function of the distribution is defined by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{(t - \mu)^2}{2\sigma^2} \right], \quad -\infty < t < +\infty, \quad (2.60)$$

where

μ and σ are the distribution parameters known as mean and standard deviation, respectively.

By substituting Equation (2.60) into Equation (2.14), we get the following equation for the cumulative distribution function:

$$F(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^t \exp \left[-\frac{(x-\mu)^2}{2\sigma^2} \right] dx. \quad (2.61)$$

Using Equation (2.60) in Equation (2.20), we get the following equation for the expected value of t :

$$E(t) = \mu. \quad (2.62)$$

2.6 Problems

1. Write an essay on the history of mathematics including probability theory.
2. Draw a Venn diagram showing two mutually exclusive sets.
3. Prove the following Boolean expression:

$$(Z + M) \cdot (Z + N) = Z + (M \cdot N) \quad (2.63)$$

where

Z , M , and N are events or sets.

4. A transportation system operation task is being performed by two independent persons X and Y . The task will not be performed correctly if either person makes an error. The probabilities of making an error by persons X and Y are 0.4 and 0.1, respectively. Calculate the probability that the task will not be accomplished successfully.
5. Write down definitions for Laplace transform and probability.
6. Obtain Laplace transform for the following function:

$$f(t) = t e^{-\lambda t}, \quad (2.64)$$

where

t is time.

λ is a constant.

7. Prove Equation (2.23).
8. Assume that the constant error rate of a transit system operator is 0.0001 errors/hour. Calculate the operator's unreliability for an 10-hour mission and mean time to human error.
9. Prove Equation (2.51).
10. Prove Equation (2.53).

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