Exponential Utility Indifference Valuation in a General Semimartingale Model

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Abstract We study the exponential utility indifference valuation of a contingent claim H when asset prices are given by a general semimartingale S. Under mild assumptions on H and S, we prove that a no-arbitrage type condition is fulfilled if and only if H has a certain representation. In this case, the indifference value can be written in terms of processes from that representation, which is useful in two ways. Firstly, it yields an interpolation expression for the indifference value which generalizes the explicit formulas known for Brownian models. Secondly, we show that the indifference value process is the first component of the unique solution (in a suitable class of processes) of a backward stochastic differential equation. Under additional assumptions, the other components of this solution are *BMO*-martingales for the minimal entropy martingale measure. This generalizes recent results by Becherer (Ann. Appl. Probab. 16:2027–2054, 2006) and Mania and Schweizer (Ann. Appl. Probab. 15:2113–2143, 2005).

Keywords Exponential utility · Indifference valuation · Minimal entropy martingale measure · BSDE · *BMO*-martingales · Fundamental entropy representation (*FER*)

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1 Introduction

One general approach to the problem of valuing contingent claims in incomplete markets is utility indifference valuation. Its basic idea is that the investor valuing a contingent claim H should achieve the same expected utility in the two cases where (1) he does not have H, or (2) he owns H but has his initial capital reduced by the amount of the *indifference value* of H. Exponential utility indifference valuation means that the utility function one uses is exponential.

This paper is dedicated to Yuri Kabanov on the occasion of his 60th birthday. We hope he likes it even if it is not short…

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Even in a concrete model, it is difficult to obtain a closed-form formula for the indifference value. The majority of existing explicit results are for Brownian settings; see for instance Frei and Schweizer [10] and the references therein. In more general situations, Becherer [2] and Mania and Schweizer [19] derive a backward stochastic differential equation (BSDE) for the indifference value process. While [19] assumes a continuous filtration, the framework in [2] has a continuous price process driven by Brownian motions and a filtration generated by these and a random measure allowing the modeling of non-predictable events.

The main contribution of this paper is to extend the above results to a setting where asset prices are given by a *general semimartingale*. We show that the exponential utility indifference value can still be written in a closed-form expression similar to that known for Brownian models, although the structure of this formula is here much less explicit. Independently from that, we establish a BSDE formulation for the dynamic indifference value process. Both of these results are based on a representation of the claim H and on the relationship between a notion of no-arbitrage, the form of the so-called minimal entropy martingale measure, and the indifference value.

As our starting point, we take the work of Biagini and Frittelli [3, 4]. Their results yield a representation of the minimal entropy martingale measure which we can use to derive a decomposition of a fixed payoff H in a similar way as in Becherer [1]. We call this decomposition, which is closely related to the minimal entropy martingale measure, the *fundamental entropy representation* of H $(FER(H))$. It is central to all our results here, because we can express the indifference value for H as a difference of terms from $FER(H)$ and $FER(0)$. We derive from this a fairly explicit formula for the indifference value by an interpolation argument, and we also establish a BSDE representation for the indifference value process. Its proof is based on the idea that the two representations $FER(H)$ and $FER(0)$ can be merged to yield a single BSDE. This direct procedure allows us to work with a general semimartingale, whereas Becherer [2] as well as Mania and Schweizer [19] use more specific models because they first prove some results for more general classes of BSDEs and then apply these to derive the particular BSDE for the indifference value. The price to pay for working in our general setting is that we must restrict the class of solutions of the BSDE to get uniqueness. Under additional assumptions, the components of the solution to the BSDE for the indifference value are again BMO-martingales for the minimal entropy martingale measure; this applies in particular to the value process of the indifference hedging strategy.

The paper is organized as follows. Section 2 lays out the model, motivates, and introduces the important notion of $FER(H)$. In Sect. 3, we prove that the existence of $FER(H)$ is essentially equivalent to an absence-of-arbitrage condition. Moreover, we develop a uniqueness result for $FER(H)$ and its relationship to the minimal entropy martingale measure. Section 4 establishes the link between the exponential indifference value of H and the two decompositions $FER(H)$ and $FER(0)$. By an interpolation argument, we derive a fairly explicit formula for the indifference value. In Sect. 5, we extend to a general filtration the BSDE representation of the indifference value by Becherer [2] and Mania and Schweizer [19]. We further provide conditions under which the components of the solution to the BSDE are *BMO*-martingales for the minimal entropy martingale measure. Section 6 rounds off with an application to a Brownian setting.

2 Motivation and Definition of $FER(H)$

We start with a probability space $(\Omega, \mathscr{F}, P)$, a finite time interval $[0,T]$ for a fixed $T > 0$ and a filtration $\mathbb{F} = (\mathscr{F}_t)_{0\le t\le T}$ satisfying the usual conditions of right-continuity and completeness. For simplicity, we assume that $\mathscr{F}_0$ is trivial and $\mathscr{F}_T = \mathscr{F}$. For a positive process Z, we use the abbreviation $Z_{t,s} := Z_s/Z_t$, $0 \le t \le s \le T$.

In our financial market, there are d risky assets whose price process $S = (S_t)_{0\le t\le T}$ is an $\mathbb{R}^d$-valued semimartingale. In addition, there is a riskless asset, chosen as numeraire, whose price is constant at 1. Our investor's risk preferences are given by an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. We always consider a fixed contingent claim H which is a real-valued $\mathscr{F}$-measurable random variable satisfying $E_P[\exp(\gamma H)] < \infty$. Expressions depending on H are introduced with an index H so we can later use them also in the absence of the claim by setting $H = 0$. However, the dependence on γ is not explicitly mentioned. We define by $\frac{\mathrm{d}P_H}{\mathrm{d}P} := \exp(\gamma H)/E_P[\exp(\gamma H)]$ a probability measure P_H on $(\Omega, \mathscr{F})$ equivalent to P. Note that $P_0 = P$. We denote by $L(S)$ the set of all $\mathbb{R}^d$-valued predictable S-integrable processes, so that $\int \vartheta \,\mathrm{d}S$ is a well-defined semimartingale for each ϑ in $L(S)$.

We always impose without further mention the following *standing assumption*, introduced by Biagini and Frittelli [3, 4] for $H = 0$. We assume that

$$\mathscr{W}_H \neq \emptyset \quad \text{and} \quad \mathscr{W}_0 \neq \emptyset, \tag{1}$$

where $\mathscr{W}_H$ is the set of *loss variables* W which satisfy the following two conditions:

(1) $W \ge 1$ P-a.s., and for every $i = 1, \ldots, d$, there exists some $\beta^i \in L(S^i)$ such that $P[\exists t \in [0,T] \text{ s.t. } \beta^i_t = 0] = 0$ and $|\int_0^t \beta^i_s \mathrm{d}S^i_s| \le W$ for all $t \in [0,T]$;
(2) $E_{P_H}[\exp(cW)] < \infty$ for all $c > 0$.

Clearly, $\mathscr{W}_H = \mathscr{W}_0$ if H is bounded. Lemma 1 at the beginning of Sect. 3 gives a less restrictive condition on H for $\mathscr{W}_H = \mathscr{W}_0$. The standing assumption (1) is automatically fulfilled if S is locally bounded since then $1 \in \mathscr{W}_H \cap \mathscr{W}_0$ by Proposition 1 of Biagini and Frittelli [3], using $P_H \approx P$. But (1) is for example also satisfied if H is bounded and $S = S^1$ is a scalar compound Poisson process with Gaussian jumps. This follows from Sect. 3.2 in Biagini and Frittelli [3]. So the model with condition (1) is a genuine generalization of the case of a locally bounded S.

To assign to H at time $t \in [0,T]$ a value based on our exponential utility function, we first fix an $\mathscr{F}_t$-measurable random variable x_t, interpreted as the investor's starting capital at time t. Then we define

$$V_t^H(x_t) := \operatorname*{ess\,sup}_{\vartheta \in \mathscr{A}_t^H} E_P\left[-\exp\left(-\gamma x_t - \gamma \int_t^T \vartheta_s \mathrm{d}S_s + \gamma H\right)\middle|\mathscr{F}_t\right], \tag{2}$$

where the set $\mathscr{A}_t^H$ of *H-admissible strategies* on $(t, T]$ consists of all processes $\vartheta I_{]\!]t,T]\!]}$ with $\vartheta \in L(S)$ and such that $\int \vartheta \mathrm{d}S$ is a Q-supermartingale for every $Q \in \mathbb{P}_H^{e,f}$; the set $\mathbb{P}_H^{e,f}$ is defined in the paragraph after the next. We recall that $x_t + \int_t^T \vartheta_s \mathrm{d}S_s$ is the investor's final wealth when starting with x_t and investing according to the self-financing strategy ϑ over $(t, T]$. Therefore, $V_t^H(x_t)$ is the maximal conditional expected utility the investor can achieve from the time-t initial capital x_t by trading during $(t, T]$ and paying out H (or receiving $-H$) at the maturity T.

The *indifference (seller) value* $h_t(x_t)$ at time t for H is implicitly defined by

$$V_t^0(x_t) = V_t^H(x_t + h_t(x_t)). \tag{3}$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with initial capital $x_t + h_t(x_t)$ but paying an additional cash-flow H at maturity T.

To define our strategies, we need the sets

$$\mathbb{P}_H^f := \left\{ Q \ll P_H \middle| I(Q|P_H) < \infty \text{ and } S \text{ is a } Q\text{-sigma-martingale} \right\},$$

$$\mathbb{P}_H^{e,f} := \left\{ Q \approx P_H \middle| I(Q|P_H) < \infty \text{ and } S \text{ is a } Q\text{-sigma-martingale} \right\},$$

where

$$I(Q|P_H) := \begin{cases} E_Q[\log \frac{\mathrm{d}Q}{\mathrm{d}P_H}] & \text{if } Q \ll P_H \\ +\infty & \text{otherwise} \end{cases}$$

denotes the relative entropy of Q with respect to P_H. Since P_H is equivalent to P, the sets $\mathbb{P}_H^f$ and $\mathbb{P}_H^{e,f}$ depend on H only through the condition $I(Q|P_H) < \infty$. By Proposition 3 and Remark 3 of Biagini and Frittelli [3], applied to P_H instead of P, there exists a unique $Q_H^E \in \mathbb{P}_H^f$ that minimizes $I(Q|P_H)$ over all $Q \in \mathbb{P}_H^f$, provided of course that $\mathbb{P}_H^f \neq \emptyset$. We call Q_H^E the *minimal H-entropy measure*, or H-MEM for short. If $\mathbb{P}_H^{e,f} \neq \emptyset$, then Q_H^E is even equivalent to P_H, i.e., $Q_H^E \in \mathbb{P}_H^{e,f}$; see Remark 2 of Biagini and Frittelli [3]. Note that the proper terminology would be "minimal H-entropy sigma-martingale measure" or H-MEσMM, but this is too long.

We briefly recall the relation between Q_H^E, Q_0^E and the indifference value $h_0(x_0)$ at time 0 to motivate the definition of $FER(H)$, which we introduce later in this section. Assume $\mathbb{P}_H^{e,f} \neq \emptyset$ and $\mathbb{P}_0^{e,f} \neq \emptyset$. The P_H-density of Q_H^E and the P-density of Q_0^E have the form

$$\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H} = c^H \exp\left(\int_0^T \zeta_s^H \mathrm{d}S_s\right) \quad \text{and} \quad \frac{\mathrm{d}Q_0^E}{\mathrm{d}P_0} = c^0 \exp\left(\int_0^T \zeta_s^0 \mathrm{d}S_s\right) \tag{4}$$

for some positive constants c^H, c^0 and processes ζ^H, ζ^0 in $L(S)$ such that $\int \zeta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$ and $\int \zeta^0 \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^f$, whence $\zeta^H \in \mathscr{A}_0^H$ and $\zeta^0 \in \mathscr{A}_0^0$. This was first shown by Kabanov and Stricker [16] in their Theorem 2.1 for a locally bounded S (and $H = 0$), and extended by Biagini and Frittelli [4] in their Theorem 1.4 to a general S for $H = 0$ (under the assumption $\mathscr{W}_0 \neq \emptyset$). By using this result also under P_H instead of P, we immediately obtain (4). It is now straightforward to calculate (and also well known—at least for locally bounded S) that for $x_0 \in \mathbb{R}$, we can write

$$\begin{aligned} V_0^H(x_0) &= \sup_{\vartheta \in \mathscr{A}_0^H} E_P\left[-\exp\left(-\gamma x_0 - \gamma \int_0^T \vartheta_s \mathrm{d}S_s + \gamma H\right)\right] \\ &= -\exp(-\gamma x_0) E_P[\exp(\gamma H)] \inf_{\vartheta \in \mathscr{A}_0^H} E_{P_H}\left[\exp\left(-\gamma \int_0^T \vartheta_s \mathrm{d}S_s\right)\right] \\ &= -\exp(-\gamma x_0) E_P[\exp(\gamma H)] \\ &\quad \times \inf_{\vartheta \in \mathscr{A}_0^H} E_{Q_H^E}\left[\frac{1}{c^H} \exp\left(\int_0^T (-\gamma \vartheta_s - \zeta_s^H) \mathrm{d}S_s\right)\right] \\ &= -\frac{\exp(-\gamma x_0) E_P[\exp(\gamma H)]}{c^H} \end{aligned} \tag{5}$$

and therefore

$$h_0(x_0) = h_0 = \frac{1}{\gamma} \log \frac{c^0 E_P[\exp(\gamma H)]}{c^H}. \tag{6}$$

In Sect. 4, we study the relation between Q_H^E, Q_0^E and $V_t^H(x_t)$, h_t for arbitrary $t \in [0, T]$. From this we can derive, on the one hand, an interpolation formula for each h_t in Sect. 4 and, on the other hand, a BSDE characterization of the process h in Sect. 5. To generalize the static relations (5), (6) to dynamic ones, we introduce a certain representation of H that we call *fundamental entropy representation of H* (*FER*(H)). Its link to the minimal H-entropy measure is elaborated in the next section. We give two different versions of this representation. The idea is that the first definition only requires some minimal conditions, whereas the second strengthens the conditions to guarantee uniqueness of the representation and ensure the identification of the H-MEM; see Proposition 2.

Definition 1 We say that *FER*(H) *exists* if there is a decomposition

$$H = \frac{1}{\gamma} \log \mathscr{E}(N^H)_T + \int_0^T \eta_s^H \mathrm{d}S_s + k_0^H, \tag{7}$$

where

(i) N^H is a local P-martingale null at 0 such that $\mathscr{E}(N^H)$ is a positive P-martingale and S is a $P(N^H)$-sigma-martingale, where $P(N^H)$ is defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathscr{E}(N^H)_T$;
(ii) η^H is in $L(S)$ and such that $\int_0^T \eta_s^H \mathrm{d}S_s \in L^1(P(N^H))$;
(iii) $k_0^H \in \mathbb{R}$ is constant.

In this case, we say that (N^H, η^H, k_0^H) is an *FER*(H). If moreover

$$\int_0^T \eta_s^H \mathrm{d}S_s \in L^1(Q) \quad \text{and} \quad E_Q\bigg[\int_0^T \eta_s^H \mathrm{d}S_s\bigg] \le 0 \quad \text{for all } Q \in \mathbb{P}_H^f$$
$$\text{and} \quad \int \eta^H \mathrm{d}S \quad \text{is a } P(N^H)\text{-martingale,} \tag{8}$$

we say that (N^H, η^H, k_0^H) is an *FER*$^\star(H)$. For any *FER*(H) (N^H, η^H, k_0^H), we set

$$k_t^H := k_0^H + \frac{1}{\gamma} \log \mathscr{E}(N^H)_t + \int_0^t \eta_s^H \mathrm{d}S_s \quad \text{for } t \in [0, T] \tag{9}$$

and call $P(N^H)$ the probability measure *associated with* (N^H, η^H, k_0^H).

Because $\mathscr{E}(N^H)$ is by (i) a positive P-martingale, the local P-martingale N^H has no negative jumps whose absolute value is 1 or more, and $P(N^H)$ is a probability measure equivalent to P. We consider two *FER*(H) (N^H, η^H, k_0^H) and $(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}_0^H)$ as equal if $\widetilde{N}^H$ and N^H are versions of each other (hence indistinguishable, since both are RCLL), $\int \widetilde{\eta}^H \mathrm{d}S$ is a version of $\int \eta^H \mathrm{d}S$, and $\widetilde{k}_0^H = k_0^H$. For future use, we note that (7) and (9) combine to give

$$H = k_t^H + \frac{1}{\gamma} \log \mathscr{E}(N^H)_{t,T} + \int_t^T \eta_s^H \mathrm{d}S_s \quad \text{for } t \in [0, T]. \tag{10}$$

The next result shows that for continuous asset prices, we can write *FER*(H) in a different (and perhaps more familiar) form. For its formulation, we need the following definition. We say that S satisfies the *structure condition* (*SC*) if

$$S^i = S_0^i + M^i + \sum_{j=1}^d \int \lambda^j \mathrm{d}\langle M^i, M^j \rangle, \quad i = 1, \ldots, d,$$

where M is a locally square-integrable local P-martingale null at 0 and λ is a predictable process such that the (final value of the) mean-variance tradeoff, $K_T = \sum_{i,j=1}^d \int_0^T \lambda_s^i \lambda_s^j \mathrm{d}\langle M^i, M^j \rangle_s = \langle \int \lambda \mathrm{d}M \rangle_T$, is almost surely finite.

Proposition 1 *Assume that S is continuous. Then a triple (N^H, η^H, k_0^H) is an FER(H) if and only if S satisfies (SC) and $\widetilde{N}^H = N^H + \int \lambda \mathrm{d}M$, $\widetilde{\eta}^H = \eta^H - \frac{1}{\gamma}\lambda$, $\widetilde{k}_0^H = k_0^H$ satisfy*

$$H = \frac{1}{\gamma} \log \mathscr{E}(\widetilde{N}^H)_T + \int_0^T \widetilde{\eta}_s^H \mathrm{d}S_s + \frac{1}{2\gamma}\Big\langle \int \lambda \mathrm{d}M \Big\rangle_T + \widetilde{k}_0^H \tag{11}$$

and

(i′) *$\widetilde{N}^H$ is a local P-martingale null at 0 and strongly P-orthogonal to each component of M, and $\mathscr{E}(\widetilde{N}^H)\mathscr{E}(-\int \lambda \mathrm{d}M)$ is a positive P-martingale*;

(ii′) $\widetilde{\eta}^H$ *is in* $L(S)$ *and such that* $\int_0^T (\widetilde{\eta}_s^H + \frac{1}{\gamma}\lambda_s)\mathrm{d}S_s$ *is* $P(N^H)$*-integrable, where* $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathscr{E}(\widetilde{N}^H)_T \mathscr{E}(-\int \lambda \mathrm{d}M)_T$;

(iii′) $\widetilde{k}_0^H \in \mathbb{R}$ *is constant.*

Proof Let first (N^H, η^H, k_0^H) be an *FER*(H). Its associated measure $P(N^H)$ is equivalent to P and S is a local $P(N^H)$-martingale since S is continuous. By Theorem 1 of Schweizer [23], S satisfies (SC) and we can write $N^H = \widetilde{N}^H - \int \lambda \mathrm{d}M$, where $\widetilde{N}^H$ is a local P-martingale null at 0 and strongly P-orthogonal to each component of M, and $\mathscr{E}(N^H) = \mathscr{E}(\widetilde{N}^H)\mathscr{E}(-\int \lambda \mathrm{d}M)$. The last equality uses that $[\widetilde{N}^H, \int \lambda \mathrm{d}M] = 0$ due to the continuity of M. Hence conditions (i)–(iii) of *FER*(H) imply (i′)–(iii′), and (7) is equivalent to (11) by (SC) and the continuity of S.

Conversely, let $(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}_0^H)$ be as in the proposition. We claim that the triple $(\widetilde{N}^H - \int \lambda \mathrm{d}M, \widetilde{\eta}^H + \frac{1}{\gamma}\lambda, \widetilde{k}_0^H)$ is an *FER*(H). Because M is a local P-martingale and $\mathscr{E}(N^H) = \mathscr{E}(\widetilde{N}^H)\mathscr{E}(-\int \lambda \mathrm{d}M)$ is the P-density process of $P(N^H)$, the process L defined by

$$L_t := M_t - \langle N^H, M \rangle_t, \quad t \in [0, T]$$

is a local $P(N^H)$-martingale by Girsanov's theorem; see for instance Theorem III.40 of Protter [21] and observe that $\langle \mathscr{E}(N^H), M \rangle = \int \mathscr{E}(N^H)_- \mathrm{d}\langle N^H, M \rangle$ exists since M is continuous like S. Because $\widetilde{N}^H$ is strongly P-orthogonal to each component of M and M is continuous, we have

$$\langle N^H, M^i \rangle = \left\langle \widetilde{N}^H - \int \lambda \mathrm{d}M, M^i \right\rangle = -\sum_{j=1}^d \int \lambda^j \mathrm{d}\langle M^j, M^i \rangle, \quad i = 1, \dots, d,$$

and so (SC) shows that $S = L + S_0$ is also a local $P(N^H)$-martingale. The other conditions of *FER*(H) are easy to check. □

Remark 1 (1) Suppose that S is continuous and satisfies (SC). If the stochastic exponential $\mathscr{E}(-\int \lambda \mathrm{d}M)$ is a P-martingale, conditions (i′) and (ii′) in Proposition 1 can be written under the probability measure $\widehat{P}$ defined by $\frac{\mathrm{d}\widehat{P}}{\mathrm{d}P} := \mathscr{E}(-\int \lambda \mathrm{d}M)_T$, which is called the *minimal local martingale measure* in the terminology of Föllmer and Schweizer [9]. This means that condition (i′) in Proposition 1 is equivalent to

(i″) $\widetilde{N}^H$ is a local $\widehat{P}$-martingale null at 0 and strongly $\widehat{P}$-orthogonal to each component of S, and $\mathscr{E}(\widetilde{N}^H)$ is a positive $\widehat{P}$-martingale,

and $P(N^H)$ can be defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}\widehat{P}} := \mathscr{E}(\widetilde{N}^H)_T$. To prove the equivalence of (i′) and (i″), first assume that $\widetilde{N}^H$ is a local P-martingale null at 0 and strongly P-orthogonal to each M^i. Then

$$\left[\widetilde{N}^H, \int \lambda \mathrm{d}M\right] = \left\langle \widetilde{N}^H, \int \lambda \mathrm{d}M \right\rangle = 0$$

by the continuity of M, and hence $\widetilde{N}^H$ is also a local $\widehat{P}$-martingale by Girsanov's theorem; see, for instance, Theorem III.40 of Protter [21]. The continuity of S, (SC) and the strong P-orthogonality of $\widetilde{N}^H$ to M entail

$$[\widetilde{N}^H, S^i] = \langle \widetilde{N}^H, M^i \rangle = 0, \quad i = 1, \dots, d,$$

implying that $\widetilde{N}^H$ is strongly $\widehat{P}$-orthogonal to each component of S. The proof of "(i$''$) $\Longrightarrow$ (i$'$)" goes analogously.

(2) Assume that S is not necessarily continuous but locally bounded and satisfies (SC) with $\lambda^i \in L^2_{loc}(M^i)$, $i = 1, \dots, d$, and let (N^H, η^H, k_0^H) be an $FER(H)$. Then we can still write $N^H = \widetilde{N}^H - \int \lambda \mathrm{d}M$ for a local P-martingale $\widetilde{N}^H$ null at 0 and strongly P-orthogonal to each component of M, by using Girsanov's theorem, (SC) and the fact that $\mathscr{E}(N^H)$ defines an equivalent local martingale measure. However, we cannot separate $\mathscr{E}(\widetilde{N}^H - \int \lambda \mathrm{d}M)$ into two factors.

3 No-arbitrage and existence of *FER*(*H*)

Theorem 1 below says that a certain notion of no-arbitrage is equivalent to the existence of $FER(H)$. It can be considered as an exponential analogue to the L^2-result of Theorem 3 in Bobrovnytska and Schweizer [5]. For a locally bounded S, the implication "$\Longrightarrow$" roughly corresponds to Proposition 2.2 of Becherer [1], who makes use of the idea to consider known results under P_H instead of P. This technique, which already appears in Delbaen et al. [6], will also be central for the proofs of our Theorem 1 and Proposition 2.

We start with a result that gives sufficient conditions for $\mathscr{W}_H \subseteq \mathscr{W}_0$ and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$ as well as for $\mathscr{W}_0 = \mathscr{W}_H$ and $\mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$. The relation between $\mathbb{P}_0^{e,f}$ and $\mathbb{P}_H^{e,f}$ will be used later, while $\mathscr{W}_0 = \mathscr{W}_H$ is helpful in applications to verify the condition (1).

Lemma 1 *If H satisfies*

$$E_P[\exp(-\varepsilon H)] < \infty \quad \textit{for some } \varepsilon > 0, \tag{12}$$

then $\mathscr{W}_H \subseteq \mathscr{W}_0$, $\mathbb{P}_0^f \subseteq \mathbb{P}_H^f$ and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$. If H satisfies

$$E_P[\exp((\gamma + \varepsilon)H)] < \infty \quad \textit{and} \quad E_P[\exp(-\varepsilon H)] < \infty$$
$$\textit{for some } \varepsilon > 0, \tag{13}$$

then $\mathscr{W}_0 = \mathscr{W}_H$, $\mathbb{P}_0^f = \mathbb{P}_H^f$ and $\mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$.

Proof We first show $\mathscr{W}_H \subseteq \mathscr{W}_0$ under (12). For $c > 0$, Hölder's inequality yields

$$E_P[\exp(cW)] = E_P\left[\exp\left(cW + \frac{\varepsilon\gamma}{\varepsilon + \gamma} H\right) \exp\left(-\frac{\varepsilon\gamma}{\varepsilon + \gamma} H\right)\right]$$

$$\leq \left(E_P\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cW+\gamma H\right)\right]\right)^{\frac{\varepsilon}{\varepsilon+\gamma}}\left(E_P[\exp(-\varepsilon H)]\right)^{\frac{\gamma}{\varepsilon+\gamma}}$$

$$= \left(E_{P_H}\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cW\right)\right]E_P[\exp(\gamma H)]\right)^{\frac{\varepsilon}{\varepsilon+\gamma}}$$

$$\times \left(E_P[\exp(-\varepsilon H)]\right)^{\frac{\gamma}{\varepsilon+\gamma}}. \tag{14}$$

Because of $E_P[\exp(\gamma H)] < \infty$ and (12), this is finite if $W \in \mathscr{W}_H$, and then $W \in \mathscr{W}_0$.

To prove $\mathscr{W}_0 = \mathscr{W}_H$ under (13), we only need to show $\mathscr{W}_0 \subseteq \mathscr{W}_H$. For $c > 0$ and $W \in \mathscr{W}_0$, we obtain similarly to (14) that

$$E_{P_H}[\exp(cW)] \leq \frac{\left(E_P[\exp((\varepsilon+\gamma)H)]\right)^{\frac{\gamma}{\varepsilon+\gamma}}}{E_P[\exp(\gamma H)]}\left(E_P\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cW\right)\right]\right)^{\frac{\varepsilon}{\varepsilon+\gamma}} < \infty$$

by (13), and hence $W \in \mathscr{W}_H$.

The remainder of the second part follows from Lemma A.1 in Becherer [1]. The proof of the rest of the first part is very similar. Indeed, (12) and the standing assumption that $E_P[\exp(\gamma H)] < \infty$ imply $E_P[\exp(\widetilde{\varepsilon}|H|)] < \infty$, where $\widetilde{\varepsilon} := \min(\varepsilon, \gamma)$. Lemma 3.5 of Delbaen et al. [6] yields

$$E_Q[\widetilde{\varepsilon}|H|] \leq I(Q|P) + \frac{1}{e}E_P[\exp(\widetilde{\varepsilon}|H|)] \quad \text{for } Q \ll P. \tag{15}$$

If $Q \in \mathbb{P}_0^f$, the right-hand side is finite, thus $E_Q[|H|] < \infty$, and we have

$$I(Q|P_H) = E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}P} - \log\frac{\mathrm{d}P_H}{\mathrm{d}P}\right] = I(Q|P) + \log E_P[\exp(\gamma H)] - \gamma E_Q[H],$$

which is finite. This shows $Q \in \mathbb{P}_H^f$, and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$ follows analogously. □

Theorem 1 *We have that*

$$\mathbb{P}_H^{e,f} \neq \emptyset \iff \mathit{FER}^\star(H) \text{ exists} \iff \mathit{FER}(H) \text{ exists}.$$

In particular, if $\mathbb{P}_0^{e,f} \neq \emptyset$ and H satisfies (12), then $\mathit{FER}^\star(H)$ exists.

Proof We first show that $\mathbb{P}_H^{e,f} \neq \emptyset$ yields the existence of $\mathit{FER}^\star(H)$. As already mentioned, $\mathbb{P}_H^{e,f} \neq \emptyset$ (and the standing assumption $\mathscr{W}_H \neq \emptyset$) imply by Proposition 3 and Remarks 2, 3 of Biagini and Frittelli [3], applied to P_H instead of P, existence and uniqueness of the H-MEM $Q_H^E \in \mathbb{P}_H^{e,f}$. Using $Q_H^E \approx P_H \approx P$, we can write

$$\frac{\mathrm{d}Q_H^E}{\mathrm{d}P} = \mathscr{E}(N^H)_T \tag{16}$$

for some local P-martingale N^H null at 0 such that $\mathscr{E}(N^H)$ is a positive P-martingale and S is a Q_H^E-sigma-martingale. Moreover, by Theorem 1.4 of Biagini and Frittelli [4], applied to P_H instead of P, we have as in (4)

$$\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H} = c^H \exp\left(\int_0^T \zeta_s^H \mathrm{d}S_s\right) \tag{17}$$

for a constant $c^H > 0$ and some ζ^H in $L(S)$ such that $\int \zeta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$. Since $\frac{\mathrm{d}P_H}{\mathrm{d}P} = \exp(\gamma H)/E_P[\exp(\gamma H)]$, comparing (17) with (16) gives

$$\mathscr{E}(N^H)_T = c_1^H \exp\left(\int_0^T \zeta_s^H \mathrm{d}S_s + \gamma H\right),$$

where $c_1^H := c^H/E_P[\exp(\gamma H)]$ is a positive constant. We thus obtain

$$H = \frac{1}{\gamma}\log \mathscr{E}(N^H)_T - \frac{1}{\gamma}\int_0^T \zeta_s^H \mathrm{d}S_s + c_2^H \quad \text{with } c_2^H := -\frac{1}{\gamma}\log c_1^H,$$

and hence $(N^H, -\frac{1}{\gamma}\zeta^H, c_2^H)$ is an $FER^\star(H)$. Note that $\int \zeta^H \mathrm{d}S$ is a $P(N^H)$-martingale because the H-MEM Q_H^E equals the probability measure $P(N^H)$ associated with $(N^H, -\frac{1}{\gamma}\zeta^H, c_2^H)$ by construction; compare (16).

To establish the equivalences of Theorem 1, it remains to show that the existence of $FER(H)$ implies $\mathbb{P}_H^{e,f} \neq \emptyset$, because every $FER^\star(H)$ is obviously an $FER(H)$. So let (N^H, η^H, k_0^H) be an $FER(H)$ and recall that its associated measure $P(N^H)$ is defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathscr{E}(N^H)_T$. We prove that $P(N^H) \in \mathbb{P}_H^{e,f}$. By condition (i) on $FER(H)$, $P(N^H)$ is a probability measure equivalent to P and S is a $P(N^H)$-sigma-martingale. To show that $P(N^H)$ has finite relative entropy with respect to P_H, we write

$$\begin{aligned}\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H} &= \frac{\mathrm{d}P(N^H)}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}P_H} = \mathscr{E}(N^H)_T \exp(-\gamma H) E_P[\exp(\gamma H)] \\ &= \exp(-\gamma k_0^H) E_P[\exp(\gamma H)] \exp\left(-\gamma \int_0^T \eta_s^H \mathrm{d}S_s\right),\end{aligned} \tag{18}$$

where the last equality is due to the decomposition (7) in $FER(H)$. This yields by (ii) of $FER(H)$ that

$$\begin{aligned} I(P(N^H)|P_H) &= E_{P(N^H)}\left[\log \frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H}\right] \\ &= -\gamma k_0^H + \log E_P[\exp(\gamma H)] - \gamma E_{P(N^H)}\left[\int_0^T \eta_s^H \mathrm{d}S_s\right] \\ &< \infty. \end{aligned}$$

Finally, the last assertion follows directly from the first part of Lemma 1. □

While the *existence* of $FER(H)$ and of $FER^\star(H)$ is equivalent by Theorem 1, the two representations are obviously different since $FER^\star(H)$ imposes more stringent conditions. The next result serves to clarify this difference.

Proposition 2 *Assume $\mathbb{P}_H^{e,f} \neq \emptyset$ and let (N^H, η^H, k_0^H) be an $FER(H)$ with associated measure $P(N^H)$. Then the following are equivalent*:

(a) *(N^H, η^H, k_0^H) is an $FER^\star(H)$, i.e., (N^H, η^H, k_0^H) satisfies* (8);
(b) *$P(N^H)$ equals the H-MEM Q_H^E, and $\int \eta^H \mathrm{d}S$ is a $P(N^H)$-martingale*;
(c) *$\int \eta^H \mathrm{d}S$ is a Q_H^E-martingale and $E_{P(N^H)}[\int_0^T \eta_s^H \mathrm{d}S_s] = 0$*;
(d) *$\int \eta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$.*

Moreover, the class of $FER^\star(H)$ consists of a singleton.

Proof Clearly, (d) implies (a), and also (c) since Q_H^E exists by Proposition 3 of Biagini and Frittelli [3], using $\mathbb{P}_H^{e,f} \neq \emptyset$ and the standing assumption $\mathscr{W}_H \neq \emptyset$. We prove "(a) $\Longrightarrow$ (b)", "(c) $\Longrightarrow$ (b)" and finally "(b) $\Longrightarrow$ (d)". The first implication goes as in the proof of Theorem 2.3 of Frittelli [11], because we have by (18) that

$$\begin{aligned}\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H} &= c_3^H \exp\left(-\gamma \int_0^T \eta_s^H \mathrm{d}S_s\right)\\ &\text{with } c_3^H := \exp(-\gamma k_0^H) E_P[\exp(\gamma H)]. \end{aligned} \tag{19}$$

The implication "(c) $\Longrightarrow$ (b)" follows from the first part of the proof of Proposition 3.2 of Grandits and Rheinländer [12], which does not use the assumption that S is locally bounded. To show "(b) $\Longrightarrow$ (d)", note that (b), (17) and (19) yield

$$c_3^H \exp\left(-\gamma \int_0^T \eta_s^H \mathrm{d}S_s\right) = c^H \exp\left(\int_0^T \zeta_s^H \mathrm{d}S_s\right) \quad P\text{-a.s.}, \tag{20}$$

where ζ^H in $L(S)$ is such that $\int \zeta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$. Taking logarithms and $P(N^H)$-expectations in (20), we obtain $c_3^H = c^H$ by using that $P(N^H) \in \mathbb{P}_H^{e,f}$ by the proof of Theorem 1. Thus $\int_0^T \eta_s^H \mathrm{d}S_s = -\frac{1}{\gamma} \int_0^T \zeta_s^H \mathrm{d}S_s$ P-a.s. and hence $\int \eta^H \mathrm{d}S = -\frac{1}{\gamma} \int \zeta^H \mathrm{d}S$ since both $\int \eta^H \mathrm{d}S$ and $\int \zeta^H \mathrm{d}S$ are $P(N^H)$-martingales. Therefore, $\int \eta^H \mathrm{d}S = -\frac{1}{\gamma} \int \zeta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$.

Theorem 1 implies the existence of $FER^\star(H)$ because $\mathbb{P}_H^{e,f} \neq \emptyset$. To show uniqueness, let (N^H, η^H, k_0^H) and $(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}_0^H)$ be two $FER^\star(H)$. Since the minimal H-entropy measure is unique by Proposition 3 of Biagini and Frittelli [3], we have from "(a) $\Longrightarrow$ (b)" that

$$\mathscr{E}(N^H)_T = \frac{\mathrm{d}Q_H^E}{\mathrm{d}P} = \mathscr{E}(\widetilde{N}^H)_T.$$

So $\mathscr{E}(\widetilde{N}^H)$ is a version of $\mathscr{E}(N^H)$ since both are P-martingales, and taking stochastic logarithms implies that $\widetilde{N}^H$ is a version of N^H. Similarly, (19) and (c) yield

$$-\gamma k_0^H + \log(E_P[\exp(\gamma H)]) = E_{Q_H^E}\left[\log \frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H}\right] \\ = -\gamma \widetilde{k}_0^H + \log(E_P[\exp(\gamma H)]),$$

thus $\widetilde{k}_0^H = k_0^H$, and therefore again from (19) that

$$\int_0^T \eta_s^H \mathrm{d}S_s = -\frac{1}{\gamma}\log\left(\frac{1}{c_3^H}\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H}\right) = \int_0^T \widetilde{\eta}_s^H \mathrm{d}S_s.$$

But both $\int \eta^H \mathrm{d}S$ and $\int \widetilde{\eta}^H \mathrm{d}S$ are Q_H^E-martingales due to (d), and so $\int \widetilde{\eta}^H \mathrm{d}S$ is a version of $\int \eta^H \mathrm{d}S$. □

Remark 2 Exploiting Proposition 3.4 of Grandits and Rheinländer [12], applied to P_H instead of P, gives a sufficient condition for $FER^\star(H)$ by using our Proposition 2. Indeed, assume that S is locally bounded and $\mathbb{P}_H^{e,f} \neq \emptyset$. If for an $FER(H)$ (N^H, η^H, k_0^H), $\int \eta^H \mathrm{d}S$ is a $BMO(P(N^H))$-martingale and $E_{P_H}[|\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H}|^{-\varepsilon}] < \infty$ for some $\varepsilon > 0$, then (N^H, η^H, k_0^H) is the $FER^\star(H)$.

Another sufficient criterion is obtained from Proposition 3.2 of Rheinländer [22] in view of our Proposition 2. Namely, if S is locally bounded and for an $FER(H)$ (N^H, η^H, k_0^H) there exists $\varepsilon > 0$ such that $E_{P_H}[\exp(\varepsilon[\int \eta^H \mathrm{d}S]_T)] < \infty$, then (N^H, η^H, k_0^H) is the $FER^\star(H)$.

While there is always at most one $FER^\star(H)$ by Proposition 2, the next example shows that there may be several $FER(H)$. This also illustrates that the uniqueness for $FER^\star(H)$ is closely related to integrability properties.

Example 1 Take two independent P-Brownian motions W and $W^\perp$, denote by $\mathbb{F}$ their P-augmented filtration and choose $d = 1$, $S = W$ and $H \equiv 0$. The MEM Q_0^E then equals P since S is a P-martingale, and $(0, 0, 0)$ is the unique $FER^\star(0)$.

To construct another $FER(0)$, choose $N^0 := W^\perp$. Then $\mathscr{E}(N^0) = \mathscr{E}(W^\perp)$ is clearly a positive P-martingale strongly P-orthogonal to $S = W$ so that condition (i) in $FER(0)$ holds. Define $P(N^0)$ as usual by $\frac{\mathrm{d}P(N^0)}{\mathrm{d}P} := \mathscr{E}(N^0)_T = \mathscr{E}(W^\perp)_T$. By Girsanov's theorem, W and $\widetilde{W}_t^\perp := W_t^\perp - t$, $0 \le t \le T$, are then $P(N^0)$-Brownian motions and we can explicitly compute

$$E_P[\log \mathscr{E}(N^0)_T] = E_P[W_T^\perp - T/2] = -T/2, \\ I(P(N^0)|P) = E_{P(N^0)}[\log \mathscr{E}(N^0)_T] = E_{P(N^0)}[\widetilde{W}_T^\perp + T/2] = T/2. \tag{21}$$

This shows that $P(N^0) \in \mathbb{P}_0^{e,f}$. Since $S = W$ is a P-Brownian motion, Proposition 1 of Emery et al. [8] now yields for every $c \in \mathbb{R}$ a process $\eta^0(c)$ in $L(S)$ such that

$$-\frac{1}{\gamma}\log \mathscr{E}\big(W^{\perp}\big)_T - c = \int_0^T \eta_s^0(c)\mathrm{d}S_s \quad P\text{-a.s.} \tag{22}$$

Because $I(P(N^0)|P) < \infty$, using the inequality $x|\log x| \le x\log x + 2e^{-1}$ shows that $\int_0^T \eta_s^0(c)\mathrm{d}S_s$ is in $L^1(P(N^0))$ so that (ii) of $FER(0)$ is also satisfied. Hence $(N^0, \eta^0(c), c)$ is an $FER(0)$, but does not coincide with $(0,0,0)$ which is the $FER^\star(0)$. To check that property (8) indeed fails, we can easily see from (21) and (22) that $\int \eta^0(c)\mathrm{d}S$ cannot be a $P(N^0)$-martingale if $c \neq -\frac{1}{2\gamma}T$. If $c = -\frac{1}{2\gamma}T$, we can simply compute, for $P \in \mathbb{P}_0^f$, that

$$E_P\left[\int_0^T \eta_s^0(c)\mathrm{d}S_s\right] = -\frac{1}{\gamma}E_P[\log \mathscr{E}(N^0)_T] + \frac{1}{2\gamma}T = \frac{1}{\gamma}T > 0.$$

We have just constructed an $FER(0)$ different from $FER^\star(0)$. Yet another $FER(0)$ can be obtained by choosing for $k \in \mathbb{R}\backslash\{0\}$ a process $\beta^0(k)$ in $L(S)$ such that

$$\int_0^{T/2} \beta_s^0(k)\mathrm{d}S_s = k \quad \text{and} \quad \int_{T/2}^T \beta_s^0(k)\mathrm{d}S_s = -k \quad P\text{-a.s.},$$

which is possible by Proposition 1 of Emery et al. [8]. Clearly, $\int_0^T \beta_s^0(k)\mathrm{d}S_s = 0$ P-a.s. and $(0, \beta^0(k), 0)$ is an $FER(0)$ (with associated measure P), which even satisfies $E_Q[\int_0^T \beta_s^0(k)\mathrm{d}S_s] = 0$ for all $Q \in \mathbb{P}_0^f$; but $\int \beta^0(k)\mathrm{d}S$ is not a P-martingale. This ends the example.

Example 1 shows that we should focus on $FER^\star(H)$ if we want to obtain good results. If S is continuous and we impose additional assumptions, the next result gives *BMO*-properties for the components of $FER^\star(H)$. This will be used later when we give a BSDE description for the exponential utility indifference value process. We first recall some definitions.

Let Q be a probability measure on $(\Omega, \mathscr{F})$ equivalent to P and $p > 1$. An adapted positive RCLL stochastic process Z is said to satisfy the *reverse Hölder inequality* $R_p(Q)$ if there exists a positive constant C such that

$$\operatorname*{ess\,sup}_{\substack{\tau \text{ stopping} \\ \text{time}}} E_Q\left[\left(\frac{Z_T}{Z_\tau}\right)^p \middle| \mathscr{F}_\tau\right] = \operatorname*{ess\,sup}_{\substack{\tau \text{ stopping} \\ \text{time}}} E_Q[(Z_{\tau,T})^p|\mathscr{F}_\tau] \le C.$$

Recall that $Z_{\tau,T} = Z_T/Z_\tau$ for a positive process Z. We say that Z satisfies the *reverse Hölder inequality* $R_{L\log L}(Q)$ if there exists a positive constant C such that

$$\operatorname*{ess\,sup}_{\substack{\tau \text{ stopping} \\ \text{time}}} E_Q[Z_{\tau,T}\log^+ Z_{\tau,T}|\mathscr{F}_\tau] \le C.$$

Z satisfies *condition* (*J*) if there exists a positive constant C such that

$$\frac{1}{C}Z_- \le Z \le C Z_-.$$

Theorem 2 *Assume that S is continuous, H is bounded and there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. Let (N^H, η^H, k_0^H) be an FER(H). Then the following are equivalent*:

(a) (N^H, η^H, k_0^H) *is the FER*$^\star(H)$;
(b) N^H *is a BMO*(P)*-martingale*, $\mathscr{E}(N^H)$ *satisfies condition* (*J*), *and* $\int \eta^H \mathrm{d}S$ *is a* $P(N^H)$*-martingale*;
(c) N^H *is a BMO*(P)*-martingale*, $\mathscr{E}(N^H)$ *satisfies condition* (*J*), *and* $\int \eta^H \mathrm{d}S$ *is a BMO*$(P(N^H))$*-martingale*;
(d) $\int \eta^H \mathrm{d}M$ *is a BMO*(P)*-martingale*, *where M is the P-local martingale part of S*;
(e) *there exists $\varepsilon > 0$ such that* $E_P[\exp(\varepsilon[\int \eta^H \mathrm{d}S]_T)] < \infty$.

The hypotheses of Theorem 2 are for instance fulfilled if H is bounded, S is continuous and satisfies (SC), and $\int \lambda \mathrm{d}M$ is a $BMO(P)$-martingale. To see this, note that $\mathscr{E}(-\int \lambda \mathrm{d}M)$ then satisfies the reverse Hölder inequality $R_p(P)$ for some $p > 1$ by Theorem 3.4 of Kazamaki [18]. The fact that there exists $k < \infty$ such that $x \log x \le k + x^p$ for all $x > 0$ now implies that $\mathscr{E}(-\int \lambda \mathrm{d}M)$ also satisfies $R_{L\log L}(P)$. Hence the minimal local martingale measure $\widehat{P}$ given by $\frac{\mathrm{d}\widehat{P}}{\mathrm{d}P} := \mathscr{E}(-\int \lambda \mathrm{d}M)_T$ is in $\mathbb{P}_0^{e,f}$ and its P-density process satisfies $R_{L\log L}(P)$.

Proof of Theorem 2 By Lemma 1, $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f} \ne \emptyset$ so that there exists an $FER(H)$ (N^H, η^H, k_0^H) by Theorem 1. Before we show that (a)–(e) are equivalent, we need some preparation. Let $\widetilde{Q}$ be a probability measure equivalent to P. Denoting by Z the P-density process of $\widetilde{Q}$ and by Y the P_H-density process of $\widetilde{Q}$, we prove that

$$Z \text{ satisfies } R_{L\log L}(P) \quad \text{if and only if} \quad Y \text{ satisfies } R_{L\log L}(P_H), \tag{23}$$

$$Z \text{ satisfies condition (J)} \quad \text{if and only if} \quad Y \text{ satisfies condition (J)}. \tag{24}$$

To that end, observe first that because H is bounded, there exists a positive constant k with $\frac{1}{k} \le \frac{\mathrm{d}P_H}{\mathrm{d}P} \le k$, which yields

$$\frac{1}{k}Z \le Y \le kZ. \tag{25}$$

For any stopping time τ, (25) implies

$$E_{P_H}[Y_{\tau,T} \log^+ Y_{\tau,T} | \mathscr{F}_\tau] \le E_P[Z_{\tau,T} \log^+(Z_{\tau,T} k^2) | \mathscr{F}_\tau],$$

and so the inequality $\log^+(ab) \le \log^+ a + \log b$ for $a > 0$ and $b \ge 1$ yields

$$E_P[Z_{\tau,T} \log^+(Z_{\tau,T} k^2) | \mathscr{F}_\tau] \le E_P[Z_{\tau,T} \log^+ Z_{\tau,T} | \mathscr{F}_\tau] + 2\log k,$$

which is bounded independently of τ if Z satisfies $R_{L\log L}(P)$. If Z satisfies condition (J) with constant C, then (25) gives

$$Y \le kZ \le kCZ_- \le k^2CY_- \quad \text{and} \quad Y \ge \frac{1}{k}Z \ge \frac{1}{kC}Z_- \ge \frac{1}{k^2C}Y_-.$$

So the "only if" part of both (23) and (24) is clear, and the "if" part is proved symmetrically.

By assumption, there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$, and so the P_H-density process of Q satisfies $R_{L\log L}(P_H)$ by (23). Because $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ is nonempty, the unique minimal H-entropy measure Q_H^E exists, and its P_H-density process also satisfies $R_{L\log L}(P_H)$ by Lemma 3.1 of Delbaen et al. [6], used for P_H instead of P. Since S is continuous, the P_H-density process of Q_H^E also satisfies condition (J) by Lemma 4.6 of Grandits and Rheinländer [12]. It follows from (23), (24) and Lemma 2.2 of Grandits and Rheinländer [12] that

$$\text{the } P\text{-density process } Z^{Q_H^E,P} \text{ of } Q_H^E \text{ satisfies } R_{L\log L}(P) \text{ and condition (J),}$$
$$\text{and the stochastic logarithm of } Z^{Q_H^E,P} \text{ is a } BMO(P)\text{-martingale.} \tag{26}$$

"(a) $\Longrightarrow$ (b)". Since (N^H, η^H, k_0^H) is the $FER^\star(H)$, Proposition 2 implies that the P-density process $Z^{Q_H^E,P}$ of Q_H^E is given by $\mathscr{E}(N^H)$ and that $\int \eta^H \mathrm{d}S$ is a $P(N^H)$-martingale. We deduce (b) from (26).

"(b) $\Longrightarrow$ (c)". We have to show that $\int \eta^H \mathrm{d}S$ is in $BMO(P(N^H))$. By conditioning (7) under $P(N^H)$ on $\mathscr{F}_\tau$ for a stopping time τ, we obtain by (b)

$$\int_0^\tau \eta_s^H \mathrm{d}S_s = -\frac{1}{\gamma}E_{P(N^H)}[\log \mathscr{E}(N^H)_T | \mathscr{F}_\tau] + E_{P(N^H)}[H|\mathscr{F}_\tau] - k_0^H,$$

and hence

$$\begin{aligned}\int_\tau^T \eta_s^H \mathrm{d}S_s = -\frac{1}{\gamma}\log \mathscr{E}(N^H)_T + \frac{1}{\gamma}E_{P(N^H)}[\log \mathscr{E}(N^H)_T | \mathscr{F}_\tau] \\ + H - E_{P(N^H)}[H|\mathscr{F}_\tau].\end{aligned}$$

By Proposition 6 of Doléans-Dade and Meyer [7], there is a $BMO(P(N^H))$-martingale $\widehat{N}^H$ with $\mathscr{E}(N^H)^{-1} = \mathscr{E}(\widehat{N}^H)$. This uses that $Z^{Q_H^E,P} = \mathscr{E}(N^H)$ satisfies condition (J) and N^H is a $BMO(P)$-martingale by (26). Since H is bounded, we get

$$\begin{aligned}&E_{P(N^H)}\left[\left|\int_\tau^T \eta_s^H \mathrm{d}S_s\right| \Big| \mathscr{F}_\tau\right] \\ &\le 2\|H\|_{L^\infty(P)} + \frac{1}{\gamma}E_{P(N^H)}\big[\big|\log \mathscr{E}(N^H)_T - E_{P(N^H)}[\log \mathscr{E}(N^H)_T|\mathscr{F}_\tau]\big|\big|\mathscr{F}_\tau\big] \\ &= 2\|H\|_{L^\infty(P)} + \frac{1}{\gamma}E_{P(N^H)}\big[\big|\log \mathscr{E}(\widehat{N}^H)_T - E_{P(N^H)}[\log \mathscr{E}(\widehat{N}^H)_T|\mathscr{F}_\tau]\big|\big|\mathscr{F}_\tau\big],\end{aligned} \tag{27}$$

and now we proceed like on page 1031 in Grandits and Rheinländer [12] to show that (27) is bounded uniformly in τ. This proves the assertion since S is continuous.

"(c) $\Longrightarrow$ (d)". Due to (26), Proposition 7 of Doléans-Dade and Meyer [7] implies that $\int \eta^H \mathrm{d}S + [\int \eta^H \mathrm{d}S, N^H]$ is a $BMO(P)$-martingale. By Proposition 1, S satisfies (SC) and $N^H = \widetilde{N}^H - \int \lambda \mathrm{d}M$ for a local P-martingale $\widetilde{N}^H$ null at 0 and strongly P-orthogonal to each component of M. Since S is continuous and satisfies (SC),

$$\Big[\int \eta^H \mathrm{d}S, N^H\Big] = \Big[\int \eta^H \mathrm{d}M, N^H\Big] = -\Big[\int \eta^H \mathrm{d}M, \int \lambda \mathrm{d}M\Big]$$
$$= -\sum_{i,j=1}^{d} \int (\eta^H)^i \lambda^j \mathrm{d}\langle M^i, M^j\rangle.$$

Hence $\int \eta^H \mathrm{d}S + [\int \eta^H \mathrm{d}S, N^H] = \int \eta^H \mathrm{d}M$ is a $BMO(P)$-martingale.

"(d) $\Longrightarrow$ (e)". We set

$$\varepsilon := \frac{1}{2\| \int \eta^H \mathrm{d}M \|^2_{BMO_2(P)}} \quad \text{and} \quad L := \sqrt{\varepsilon} \int \eta^H \mathrm{d}M.$$

Clearly, L is like $\int \eta^H \mathrm{d}M$ a continuous $BMO(P)$-martingale and we have that $\|L\|_{BMO_2(P)} = 1/\sqrt{2} < 1$. Since S is continuous, the John-Nirenberg inequality (see Theorem 2.2 of Kazamaki [18]) yields

$$E_P\Big[\exp\Big(\varepsilon\Big[\int \eta^H \mathrm{d}S\Big]_T\Big)\Big] = E_P[\exp([L]_T)] \le \frac{1}{1 - \|L\|^2_{BMO_2(P)}} < \infty.$$

"(e) $\Longrightarrow$ (a)". This is based on the same idea as the proof of Proposition 3.2 of Rheinländer [22]. Lemma 3.5 of Delbaen et al. [6] yields

$$E_Q\Big[\varepsilon\Big[\int \eta^H \mathrm{d}S\Big]_T\Big] \le I(Q|P_H) + \frac{1}{e} E_{P_H}\Big[\exp\Big(\varepsilon\Big[\int \eta^H \mathrm{d}S\Big]_T\Big)\Big] < \infty$$

for any $Q \in \mathbb{P}^f_H$ because H is bounded and (e) holds. So $[\int \eta^H \mathrm{d}S]_T$ is Q-integrable and thus the local Q-martingale $\int \eta^H \mathrm{d}S$ is a square-integrable Q-martingale for any $Q \in \mathbb{P}^f_H$. This concludes the proof in view of Proposition 2. □

4 Relating $FER^\star(H)$ and $FER^\star(0)$ to the Indifference Value

In this section, we establish the connection between $FER^\star(H)$, $FER^\star(0)$ and the indifference value process h. We then derive and study an interpolation formula for h. Throughout this section, we assume that

$$\mathbb{P}^{e,f}_H \neq \emptyset \quad \text{and} \quad \mathbb{P}^{e,f}_0 \neq \emptyset,$$

and we denote by (N^H, η^H, k_0^H) and (N^0, η^0, k_0^0) the unique $FER^\star(H)$ and $FER^\star(0)$ with associated measures $P(N^H) = Q_H^E$ and $P(N^0) = Q_0^E$, respectively.

Our first result expresses the maximal expected utility and the indifference value in terms of the given $FER^\star(H)$ and $FER^\star(0)$. For a locally bounded S, this is very similar to Becherer [1]; see in particular there Propositions 2.2 and 3.5 and the discussion on page 12 at the end of Sect. 3. Indeed, the main differences are that the representation in [1] is given in terms of certainty equivalents instead of maximal conditional expected utilities and S is locally bounded; but the results are the same.

Theorem 3 *V^H, V^0 and h are well defined and, for any $t \in [0,T]$ and any $\mathscr{F}_t$-measurable random variable x_t, we have*

$$V_t^H(x_t) = -\exp(-\gamma x_t + \gamma k_t^H) \tag{28}$$

and

$$h_t(x_t) = h_t = k_t^H - k_t^0, \tag{29}$$

where k_t^H (and k_t^0, with the obvious adaptations) are defined in (9).

Proof Let us first write (2) as

$$V_t^H(x_t) = -\exp(-\gamma x_t) \operatorname*{ess\,inf}_{\vartheta \in \mathscr{A}_t^H} \varphi_t^H(\vartheta) \tag{30}$$

with the abbreviation

$$\varphi_t^H(\vartheta) := E_P\left[\exp\left(-\gamma \int_t^T \vartheta_s \mathrm{d}S_s + \gamma H\right)\middle|\mathscr{F}_t\right].$$

Because (N^H, η^H, k_0^H) is the $FER^\star(H)$, $\varphi_t^H(\vartheta)$ can be written by (10) as

$$\begin{aligned}\varphi_t^H(\vartheta) &= \exp(\gamma k_t^H) E_P\left[\mathscr{E}(N^H)_{t,T} \exp\left(\gamma \int_t^T (\eta_s^H - \vartheta_s)\mathrm{d}S_s\right)\middle|\mathscr{F}_t\right] \\ &= \exp(\gamma k_t^H) E_{P(N^H)}\left[\exp\left(\gamma \int_t^T (\eta_s^H - \vartheta_s)\mathrm{d}S_s\right)\middle|\mathscr{F}_t\right],\end{aligned} \tag{31}$$

using Bayes' formula. Since $P(N^H) = Q_H^E \in \mathbb{P}_H^{e,f}$ and $\int \vartheta \mathrm{d}S$ is a Q-supermartingale and $\int \eta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$, we have

$$E_{P(N^H)}\left[\int_t^T (\eta_s^H - \vartheta_s)\mathrm{d}S_s\middle|\mathscr{F}_t\right] \geq 0$$

which implies $\varphi_t^H(\vartheta) \geq \exp(\gamma k_t^H)$ by Jensen's inequality and (31). On the other hand, the choice

$$\vartheta_s^\star := \eta_s^H, \quad s \in (t,T], \tag{32}$$

gives $\varphi_t^H(\vartheta^\star) = \exp(\gamma k_t^H)$ by (31). Because $\int \vartheta^\star \mathrm{d}S = \int \eta^H \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$, $\vartheta^\star$ is in $\mathscr{A}_t^H$, and (28) now follows from (30).

By the same reasoning as for (28), we obtain

$$V_t^0(x_t) = -\exp(-\gamma x_t + \gamma k_t^0).$$

Solving the implicit equation (3) for $h_t(x_t)$ then immediately leads to (29). □

The proof of Theorem 3, especially (32), gives an interpretation for the $FER^\star(H)$. An investor who must pay out the claim H at time T uses, under exponential utility preferences, the decomposition (7). The portion of H that he hedges by trading in S is $\int_0^T \eta_s^H \mathrm{d}S_s$, whereas $\frac{1}{\gamma} \log \mathscr{E}(N^H)_T$ remains unhedged. Moreover, the proof of Theorem 3 shows that for $t \in [0,T]$ and an $\mathscr{F}_t$-measurable x_t, the value of $V_t^H(x_t)$ is not affected if we restrict the set $\mathscr{A}_t^H$ to those $\vartheta \in \mathscr{A}_t^H$ such that $\int \vartheta \mathrm{d}S$ is not only a Q-supermartingale, but a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$.

Proposition 3 *Assume that H satisfies (12). Then for any $Q \in \mathbb{P}_0^f$ and $t \in [0,T]$,*

$$h_t = E_Q[H|\mathscr{F}_t] - \frac{1}{\gamma} E_Q\left[\log \frac{\mathscr{E}(N^H)_{t,T}}{\mathscr{E}(N^0)_{t,T}} \middle| \mathscr{F}_t\right]. \tag{33}$$

In particular,

$$h_0 = E_Q[H] + \frac{1}{\gamma}\big(I(Q|Q_H^E) - I(Q|Q_0^E)\big). \tag{34}$$

The decomposition (34) of the indifference value h_0 can be described as follows. The first term, $E_Q[H]$, is the expected payoff under a measure $Q \in \mathbb{P}_0^f$. This is linear in the number of claims. The second term is a nonlinear correction term or safety loading. It can be interpreted as the difference of the distances from Q_H^E and Q_0^E to Q (although $I(\cdot|\cdot)$ is not a metric). This correction term is not based on all of H, but only on the processes N^H and N^0 from the $FER^\star(H)$ and $FER^\star(0)$, i.e., on the unhedged parts of H and 0, respectively. A similar decomposition also appears for indifference pricing under quadratic preferences; see Schweizer [24].

If H satisfies (12), then the indifference value process h is a Q_0^E-supermartingale. In fact, Jensen's inequality and (33) with $Q = Q_0^E$ yield $h_t \geq E_{Q_0^E}[H|\mathscr{F}_t]$ P-a.s. for $t \in [0,T]$ and so $h_t^- \in L^1(Q_0^E)$ since H is Q_0^E-integrable due to (12); compare (15). Moreover, $Z := \mathscr{E}(N^H)/\mathscr{E}(N^0)$ is a Q_0^E-martingale as it is the Q_0^E-density process of Q_H^E. Thus $\log Z$ has the Q_0^E-supermartingale property by Jensen's inequality, and so has h since $h_t = E_{Q_0^E}[H|\mathscr{F}_t] - \frac{1}{\gamma} E_{Q_0^E}[\log Z_T|\mathscr{F}_t] + \frac{1}{\gamma}\log Z_t$ for $t \in [0,T]$ by (33). Now $E_{Q_0^E}[h_t] \leq h_0 < \infty$ shows that h_t is Q_0^E-integrable for every $t \in [0,T]$.

Proof of Proposition 3 Since $Q \in \mathbb{P}_0^f \subseteq \mathbb{P}_H^f$ by Lemma 1, $\int \eta^H \mathrm{d}S$ is a Q-martingale by Proposition 2. Moreover, H is Q-integrable due to (12); compare (15). From

(10), we thus obtain for $t \in [0, T]$ that

$$k_t^H = E_Q\left[H - \frac{1}{\gamma}\log \mathscr{E}(N^H)_{t,T}\middle|\mathscr{F}_t\right]. \tag{35}$$

Plugging (35) and the analogous expression for k_t^0 into (29) leads to (33).

To prove (34), we first show that $I(Q|Q_0^E)$ is finite. We can write

$$\begin{aligned} I(Q|Q_0^E) &= E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}P} + \log\frac{\mathrm{d}P}{\mathrm{d}Q_0^E}\right] \\ &= I(Q|P) - E_Q[\log \mathscr{E}(N^0)_T] < \infty \end{aligned} \tag{36}$$

because $Q \in \mathbb{P}_0^f$ and $-E_Q[\log \mathscr{E}(N^0)_T] = \gamma k_0^0$ by (35) for $H = 0$ and $t = 0$. Moreover, $Q \ll P \approx Q_H^E$ gives $\frac{\mathrm{d}Q}{\mathrm{d}P} > 0$ Q-a.s. and thus from

$$\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} = \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}Q_H^E} = \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{1}{\mathscr{E}(N^H)_T} \quad Q\text{-a.s.}$$

that

$$-\log \mathscr{E}(N^H)_T = \log\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} - \log\frac{\mathrm{d}Q}{\mathrm{d}P} \quad Q\text{-a.s.},$$

and analogously for 0 instead of H. Hence

$$\begin{aligned} E_Q\left[-\log\frac{\mathscr{E}(N^H)_T}{\mathscr{E}(N^0)_T}\right] &= E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} - \log\frac{\mathrm{d}Q}{\mathrm{d}Q_0^E}\right] \\ &= I(Q|Q_H^E) - I(Q|Q_0^E), \end{aligned}$$

where we have used (36) for the last equality. Now (34) follows from (33). □

We next come to the announced interpolation formula for the indifference value.

Theorem 4 *Let $Q \in \mathbb{P}_H^{e,f}$ and φ in $L(S)$ be such that $\int \varphi \mathrm{d}S$ is a Q- and Q_H^E-martingale. Fix $t \in [0, T]$, denote by Z the P-density process of Q, set*

$$\Psi_t^H := \frac{\exp(\gamma H + \int_t^T \varphi_s \mathrm{d}S_s)}{Z_{t,T}} \tag{37}$$

and assume that Ψ_t^H and $\log\Psi_t^H$ are Q-integrable. Then there exists an $\mathscr{F}_t$-measurable random variable $\delta_t^H : \Omega \to [1, \infty]$ such that for almost all $\omega \in \Omega$,

$$k_t^H(\omega) = \frac{1}{\gamma}\log(E_Q[|\Psi_t^H|^{1/\delta}|\mathscr{F}_t](\omega))^{\delta}|_{\delta=\delta_t^H(\omega)}, \tag{38}$$

where

$$\log(E_Q[|\Psi_t^H|^{1/\delta}|\mathscr{F}_t](\omega))^\delta|_{\delta=\infty} := \lim_{\delta\to\infty}\log(E_Q[|\Psi_t^H|^{1/\delta}|\mathscr{F}_t](\omega))^\delta$$
$$= E_Q[\log\Psi_t^H|\mathscr{F}_t](\omega) \tag{39}$$

for almost all $\omega\in\Omega$.

In view of $h_t = k_t^H - k_t^0$ by Theorem 3, (38) gives us a quasi-explicit formula for the exponential utility indifference value if H is bounded and if we can find a measure $Q\in\mathbb{P}_0^{e,f}$ such that the corresponding Ψ_t^0 given in (37) and $\log\Psi_t^0$ are Q-integrable for some predictable φ such that $\int\varphi\mathrm{d}S$ is a Q-, Q_0^E- and Q_H^E-martingale. For $t=0$, one possible choice is the minimal 0-entropy measure Q_0^E which is by (19) and Proposition 2 of the form $\frac{\mathrm{d}Q_0^E}{\mathrm{d}P} = c_3^0\exp(\int_0^T\zeta_s^0\mathrm{d}S_s)$ for a constant c_3^0 and a predictable process ζ^0 such that $\int\zeta^0\mathrm{d}S$ is a Q-martingale for every $Q\in\mathbb{P}_0^f$. One disadvantage of this choice is that Q_0^E is in general unknown; a second is that we still need to find some φ, and we know almost nothing about the potential candidate ζ^0. In Corollary 1, we give conditions under which the explicitly known minimal local martingale measure $\widehat{P}$ satisfies the assumptions of Theorem 4.

Proof of Theorem 4 From (10) and (37), we obtain via $\frac{\mathrm{d}Q_H^E}{\mathrm{d}P} = \mathscr{E}(N^H)_T$ and Bayes' formula that

$$\begin{aligned}\exp(-\gamma k_t^H)E_Q[\Psi_t^H|\mathscr{F}_t] &= E_Q\left[\frac{\mathscr{E}(N^H)_{t,T}}{Z_{t,T}}\exp\left(\int_t^T(\varphi_s+\gamma\eta_s^H)\mathrm{d}S_s\right)\middle|\mathscr{F}_t\right]\\ &= E_{Q_H^E}\left[\exp\left(\int_t^T(\varphi_s+\gamma\eta_s^H)\mathrm{d}S_s\right)\middle|\mathscr{F}_t\right]\\ &\geq \exp\left(E_{Q_H^E}\left[\int_t^T(\varphi_s+\gamma\eta_s^H)\mathrm{d}S_s\middle|\mathscr{F}_t\right]\right)\\ &= 1\end{aligned} \tag{40}$$

by Jensen's inequality and because $\int\varphi\mathrm{d}S$ and $\int\eta^H\mathrm{d}S$ are Q_H^E-martingales. Hence

$$k_t^H \leq \frac{1}{\gamma}\log E_Q[\Psi_t^H|\mathscr{F}_t]. \tag{41}$$

On the other hand, (35), (37) and Jensen's inequality yield

$$\begin{aligned}\gamma k_t^H &= E_Q[\gamma H - \log\mathscr{E}(N^H)_{t,T}|\mathscr{F}_t]\\ &= E_Q\left[\log\Psi_t^H - \log\frac{\mathscr{E}(N^H)_{t,T}}{Z_{t,T}}\middle|\mathscr{F}_t\right]\\ &\geq E_Q[\log\Psi_t^H|\mathscr{F}_t].\end{aligned} \tag{42}$$

Consider the stochastic process $f(\cdot,\cdot):[1,\infty)\times\Omega\to\mathbb{R}$ defined by

$$f(\delta,\omega):=\log(E_Q[|\Psi_t^H|^{\frac{1}{\delta}}|\mathscr{F}_t](\omega))^{\delta},\quad (\delta,\omega)\in[1,\infty)\times\Omega.$$

Because $|\Psi_t^H|^{1/\delta}\le 1+\Psi_t^H\in L^1(Q)$ for all $\delta\in[1,\infty)$, Lebesgue's dominated convergence theorem and Jensen's inequality for conditional expectations allow us to choose a version of f which is continuous and nonincreasing in δ for all fixed $\omega\in\Omega$, so that by monotonicity, the limit $f(\infty,\omega):=\lim_{\delta\to\infty}f(\delta,\omega)$ exists for all $\omega\in\Omega$. We next show that

$$f(\infty,\omega)=E_Q[\log\Psi_t^H|\mathscr{F}_t](\omega)\quad\text{for almost all }\omega\in\Omega. \tag{43}$$

To ease the notation, we define $g(\cdot,\cdot):[1,\infty)\times\Omega\to\mathbb{R}$ by

$$g(\delta,\omega):=(\exp(f(\delta,\omega)))^{\frac{1}{\delta}}=E_Q[|\Psi_t^H|^{\frac{1}{\delta}}|\mathscr{F}_t](\omega),\quad(\delta,\omega)\in[1,\infty)\times\Omega$$

so that $f(\delta,\omega)=\delta\log g(\delta,\omega)$. Again since $|\Psi_t^H|^{1/\delta}\le 1+\Psi_t^H\in L^1(Q)$ for all $\delta\in[1,\infty)$, dominated convergence gives

$$\lim_{n\to\infty}g(n,\omega)=1\quad\text{for almost all }\omega\in\Omega. \tag{44}$$

For $x>1/2$ we have $x-1\ge\log x\ge x-1-|x-1|^2$, from which we obtain by (44) that for almost all $\omega\in\Omega$, there exists $n_0(\omega)\in\mathbb{N}$ such that

$$n(g(n,\omega)-1)\ge f(n,\omega)\ge n(g(n,\omega)-1)-n|g(n,\omega)-1|^2,\quad n\ge n_0(\omega). \tag{45}$$

In view of (44) and (45), we get (43) if we show that

$$\lim_{n\to\infty}n(g(n,\omega)-1)=E_Q[\log\Psi_t^H|\mathscr{F}_t](\omega)\quad\text{for almost all }\omega\in\Omega. \tag{46}$$

But (46) follows from Lebesgue's convergence theorem and

$$\lim_{n\to\infty}n(|\Psi_t^H|^{\frac{1}{n}}-1)=\lim_{n\to\infty}n\left(\exp\left(\frac{1}{n}\log\Psi_t^H\right)-1\right)=\log\Psi_t^H\quad P\text{-a.s.}$$

if we show that $n||\Psi_t^H|^{1/n}-1|$, $n\in\mathbb{N}$, is dominated by a Q-integrable random variable. Due to $e^x-1\ge x$ for $x\in\mathbb{R}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}x(a^{\frac{1}{x}}-1)=a^{\frac{1}{x}}\left(1-\frac{1}{x}\log a\right)-1\le a^{\frac{1}{x}}\exp\left(-\frac{1}{x}\log a\right)-1=0$$

for $a>0$ and $x>0$, it follows for $a=\Psi_t^H$ that

$$\log\Psi_t^H\le n\left(\exp\left(\frac{1}{n}\log\Psi_t^H\right)-1\right)\le\Psi_t^H-1,\quad n\in\mathbb{N}.$$

This gives $n||\Psi_t^H|^{1/n}-1|\le|\log\Psi_t^H|+\Psi_t^H\in L^1(Q)$, $n\in\mathbb{N}$, and proves (43).

Combining (41), (42) and (43) yields $f(\infty,\omega) \le \gamma k_t^H(\omega) \le f(1,\omega)$ for almost all $\omega \in \Omega$. By the intermediate value theorem, the set

$$\Delta(\omega) := \{\delta \in [1,\infty] \mid f(\delta,\omega) = \gamma k_t^H(\omega)\}$$

is thus nonempty for almost all $\omega \in \Omega$. Define $\delta_t^H : \Omega \to [1,\infty]$ by

$$\delta_t^H(\omega) := \sup \Delta(\omega), \quad \omega \in \Omega, \tag{47}$$

setting $\delta_t^H := 1$ on the P-null set $\{\omega \in \Omega | \Delta(\omega) = \emptyset\}$. By continuity of f in δ, $\Delta(\omega)$ is closed in $\mathbb{R} \cup \{+\infty\}$ for all $\omega \in \Omega$, and we get for almost all $\omega \in \Omega$ that

$$f(\delta_t^H(\omega),\omega) = \gamma k_t^H(\omega). \tag{48}$$

It remains to prove that the mapping $\omega \mapsto \delta_t^H(\omega)$ is $\mathscr{F}_t$-measurable. Because f is nonincreasing and due to (47) and (48), we have for any $a \in [1,\infty]$ that

$$\begin{aligned}\{\omega \in \Omega | \delta_t^H(\omega) < a\} &= \{\omega \in \Omega | f(\delta_t^H(\omega),\omega) > f(a,\omega)\}\\ &= \{\omega \in \Omega | \gamma k_t^H(\omega) > f(a,\omega)\}\\ &= \bigcup_{q\in\mathbb{Q}} \big(\{\omega \in \Omega | \gamma k_t^H(\omega) > q\} \cap \{\omega \in \Omega | q > f(a,\omega)\}\big)\end{aligned}$$

up to a P-null set. The last set is in $\mathscr{F}_t$ because k_t^H and $f(a,\cdot)$ for fixed $a \in [1,\infty]$ are $\mathscr{F}_t$-measurable random variables. Since $\mathscr{F}_t$ is complete, $\{\omega \in \Omega | \delta_t^H(\omega) < a\}$ is in $\mathscr{F}_t$ for every $a \in \mathbb{R} \cup \{+\infty\}$, and so δ_t^H is $\mathscr{F}_t$-measurable. □

The next result provides a simplified version of Theorem 4 based on the use of the minimal local martingale measure $\widehat{P}$.

Corollary 1 *Fix $t \in [0,T]$ and assume that H is bounded and S satisfies (SC). Suppose further that $\widehat{P}$ given by $\frac{\mathrm{d}\widehat{P}}{\mathrm{d}P} := \mathscr{E}(-\int \lambda \mathrm{d}M)_T$ is in $\mathbb{P}_0^{e,f}$, that $\int \lambda \mathrm{d}S$ is a $\widehat{P}$-, Q_0^E- and Q_H^E-martingale, and that the random variable*

$$\exp\Big(-\Big\langle \int \lambda \mathrm{d}M\Big\rangle + \frac{1}{2}\Big[\int \lambda \mathrm{d}M\Big]^c\Big)_{t,T} \prod_{t<s\le T} \frac{e^{-\lambda_s \cdot \Delta M_s}}{1-\lambda_s \cdot \Delta M_s}$$

and its logarithm are $\widehat{P}$-integrable. Then there exist $\mathscr{F}_t$-measurable random variables $\delta_t^0, \delta_t^H : \Omega \to [1,\infty]$ such that for almost all $\omega \in \Omega$,

$$\begin{aligned}h_t(\omega) = &\frac{1}{\gamma}\log(E_{\widehat{P}}[|\Psi_t^H|^{1/\delta}|\mathscr{F}_t](\omega))^{\delta}\big|_{\delta=\delta_t^H(\omega)}\\ &-\frac{1}{\gamma}\log(E_{\widehat{P}}[|\Psi_t^0|^{1/\delta'}|\mathscr{F}_t](\omega))^{\delta'}\big|_{\delta'=\delta_t^0(\omega)},\end{aligned}$$

where we use the convention (39) *and the definition*

$$\Psi_t^H := \frac{\exp(\gamma H - \int_t^T \lambda_s \mathrm{d}S_s)}{\mathscr{E}(-\int \lambda \mathrm{d}M)_{t,T}} = \frac{e^{\gamma H} \exp(-\int \lambda \mathrm{d}S)_{t,T}}{\mathscr{E}(-\int \lambda \mathrm{d}M)_{t,T}}. \tag{49}$$

Proof We only need to check that Ψ_t^0, Ψ_t^H given by (49) and $\log \Psi_t^0$, $\log \Psi_t^H$ are $\widehat{P}$-integrable as the result then follows from Theorems 3 and 4 with the choice $Q := \widehat{P}$ and $\varphi := -\lambda$. Using the formula for the stochastic exponential and (SC), we get

$$\Psi_t^0 = \exp\Big(-\Big\langle \int \lambda \mathrm{d}M \Big\rangle + \frac{1}{2}\Big[\int \lambda \mathrm{d}M\Big]^c\Big)_{t,T} \prod_{t<s\le T} \frac{e^{-\lambda_s \cdot \Delta M_s}}{1 - \lambda_s \cdot \Delta M_s},$$

and thus $\Psi_t^0, \log \Psi_t^0 \in L^1(\widehat{P})$ by assumption. The same is true for Ψ_t^H because H is bounded by assumption. □

To the best of our knowledge, results like Theorem 4 and Corollary 1 have not been available in the literature so far. A closed-form expression for the exponential utility indifference value has been known only in specific cases when the asset prices are modeled by continuous semimartingales; see for example [10] for explicit expressions of the indifference value in two Brownian settings. There the adapted process δ^H, called the *distortion power*, is closely related to the instantaneous correlation between the driving Brownian motions. The model in [10] consists of a risk-free bank account and a stock $S = S^1$ driven by a Brownian motion W. The claim H depends on another Brownian motion Y which has a time-dependent and fairly general instantaneous stochastic correlation ϱ with W, with $|\varrho|$ uniformly bounded away from 1. Theorem 2 of [10] proves that the indifference value is of the form of Corollary 1 above, with δ_t^H and δ_t^0 taking values between

$$\underline{\delta}_t := \inf_{s\in[t,T]} \frac{1}{\|1 - |\varrho_s|^2\|_{L^\infty(P)}} \quad \text{and} \quad \overline{\delta}_t := \sup_{s\in[t,T]} \Big\| \frac{1}{1 - |\varrho_s|^2} \Big\|_{L^\infty(P)}.$$

For small $|\varrho|$ (uniformly in s, in the L^∞-norm), the claim H is almost unhedgeable and $1/\delta^H$ is nearly 1, whereas for $|\varrho|$ close to 1, the claim H is well hedgeable and $1/\delta^H$ is nearly 0. So in that Brownian model, $1/\delta^H$ is closely related to some kind of distance of H from being attainable or hedgeable. In the subsequent discussion, we extend this idea to a more general setting, while we come back to the Brownian model in Sect. 6.

Consider the setting of Corollary 1 where S is (in addition) continuous and satisfies (SC), and H is bounded. Then the P-martingale part M of S is also continuous and the mean-variance tradeoff process $K = \langle \int \lambda \mathrm{d}M\rangle = \langle \int \lambda \mathrm{d}S\rangle$ is P-a.s. finite by (SC). The quantity Ψ_t^H from (49) then reduces to $\Psi_t^H = \exp(\gamma H - \frac{1}{2}(K_T - K_t))$, and the assumptions of Corollary 1 are satisfied if K_T is bounded, because $\int \lambda \mathrm{d}M$ is then a $BMO(P)$-martingale. If we now even suppose that K_T is deterministic, the indifference value at time 0 simplifies to

$$h_0 = \frac{1}{\gamma}\log(E_{\widehat{P}}[\exp(\gamma H/\delta)])^{\delta}|_{\delta=\delta_0^H} \tag{50}$$

by Corollary 1. If $\delta_0^H < \infty$, we can write

$$h_0 = -\widetilde{U}_H^{-1}(E_{\widehat{P}}[\widetilde{U}_H(-H)]), \quad \text{where } \widetilde{U}_H(x) := -\exp(-\gamma x/\delta_0^H),\ x \in \mathbb{R},$$

which means that $-h_0$ is a certainty equivalent of $-H$. Note, however, that this is done under $\widehat{P}$, not P, and with respect to the utility function $\widetilde{U}_H$, not U, where $\widetilde{U}_H$ depends itself on the claim H. If $\delta_0^H = 1$, then $\widetilde{U}_H$ and U coincide and H is valued by the U-certainty equivalent under $\widehat{P}$. Moreover, (38) shows that we then must have equality in (40) for $t = 0$, which implies that $\int_0^T(\gamma\eta_s^H - \lambda_s)\mathrm{d}S_s$ is deterministic, hence $\int(\gamma\eta^H - \lambda)\mathrm{d}S = 0$. In other words, the equivalent formulation (11) of $FER(H)$ in Proposition 1 simplifies in this case to

$$H = \frac{1}{\gamma}\log\mathscr{E}(\widetilde{N}^H)_T + \frac{1}{2\gamma}K_T + k_0^H,$$

which means that H consists only of a constant plus an unhedged term. This may be interpreted as saying that H has maximal distance to attainability. On the opposite extreme, the case $\delta_0^H = \infty$ leads by (50) and (39) (and still under the same assumptions) to $h_0 = E_{\widehat{P}}[H]$. Hence for $\delta_0^H = \infty$, we get a familiar no-arbitrage value for H. In this case, (38) and (39) show that we must have equality in (42) for $t = 0$; hence $\mathscr{E}(N^H) = \mathscr{E}(-\int\lambda\mathrm{d}M)$ and thus (11) simplifies to

$$H = \int_0^T \widetilde{\eta}_s^H \mathrm{d}S + \frac{1}{2\gamma}K_T + k_0^H,$$

showing that H is attainable. Summing up, we can interpret $1/\delta^H$ as the distance of H from being attainable; for $1/\delta^H = 0$ (convention: $1/\infty = 0$), the distance is minimal, whereas for $1/\delta^H = 1$, it is maximal. The following remark shows how this idea can be made mathematically more precise.

Remark 3 Assume that S is continuous, satisfies (SC) and that $K_T = \langle\int\lambda\mathrm{d}M\rangle_T$ is bounded, but not necessarily deterministic. By Theorem 4 and Corollary 1, we can attribute to any $H \in L^\infty(P)$ a number $\delta(H) := \delta_0^H$ in $[1,\infty]$ uniquely defined via (47) with $Q = \widehat{P}$ and $\varphi = -\lambda$. Defining for $G, H \in L^\infty(P)$

$$G \sim H \ :\Longleftrightarrow\ \delta\Big(G + \frac{1}{2\gamma}K_T\Big) = \delta\Big(H + \frac{1}{2\gamma}K_T\Big)$$

gives an equivalence relation on $L^\infty(P)$. We denote by $D := L^\infty(P)/\sim$ the set of its equivalence classes and associate to each equivalence class a representative. We further define the mapping $d : D \times D \to [0, 1]$ for $G, H \in D$ by

$$d(G, H) := \left| \frac{1}{\delta(G + \frac{1}{2\gamma} K_T)} - \frac{1}{\delta(H + \frac{1}{2\gamma} K_T)} \right|.$$

Clearly, d is a metric on D. A claim $G \in L^\infty(P)$ is called ($\widehat{P}$-)*attainable* if it can be written as $G = E_{\widehat{P}}[G] + \int_0^T \beta_s \mathrm{d}S_s$ for a predictable process β such that $\int \beta \mathrm{d}S$ is a $\widehat{P}$-martingale, which is then even a $BMO(\widehat{P})$-martingale. If G is attainable, the *FER** of $G + \frac{1}{2\gamma} K_T$ equals $(-\int \lambda \mathrm{d}M, \beta + \frac{1}{\gamma}\lambda, E_{\widehat{P}}[G])$, and so the term $\log \frac{\mathscr{E}(N^H)_T}{\mathscr{E}(-\int \lambda \mathrm{d}M)_T}$ vanishes identically. This implies $\delta(G + \frac{1}{2\gamma} K_T) = \infty$ by the proof of Theorem 4, hence $G \sim 0$. Therefore,

$$d(0, H) = \frac{1}{\delta(H + \frac{1}{2\gamma} K_T)}$$

is a distance of $H \in L^\infty(P)$ from attainability.

The maximal value of $d(0, \cdot)$ depends on the diversity of the filtration $\mathbb{F}$. If S has the predictable representation property in $\mathbb{F}$ in the sense that any $H \in L^\infty(P)$ is attainable (as above), then $\sim$ has only one equivalence class and $d \equiv 0$. On the other hand, suppose that there exists a nondeterministic local $\widehat{P}$-martingale N null at 0 and strongly $\widehat{P}$-orthogonal to each component of S such that $\mathscr{E}(N)$ is a $\widehat{P}$-martingale bounded away from zero and infinity. The maximal distance to attainability is then attained by $\frac{1}{\gamma} \log \mathscr{E}(N)_T$ since $d(0, \frac{1}{\gamma} \log \mathscr{E}(N)_T) = 1$.

5 A BSDE Characterization of the Indifference Value Process

In this section, we prove that the indifference value process h is (the first component of) the unique solution, in a suitable class of processes, of a backward stochastic differential equation (BSDE). This result is similar to Becherer [2] and Mania and Schweizer [19], but obtained here in a general (not even locally bounded) semimartingale model.

We assume throughout this section that

$$\mathbb{P}_0^{e,f} \neq \emptyset$$

and denote by Q_0^E the minimal 0-entropy measure. Let us consider the BSDE

$$\Gamma_t = \Gamma_0 + \frac{1}{\gamma} \log \mathscr{E}(L)_t + \int_0^t \psi_s \mathrm{d}S_s, \quad t \in [0, T] \tag{51}$$

with the boundary condition

$$\Gamma_T = H. \tag{52}$$

We introduce three different notions of solutions to (51), (52).

Definition 2 We say that the triple (Γ, ψ, L) is a *solution* of (51), (52) if

(Si) Γ is a real-valued semimartingale;
(Sii) ψ is in $L(S)$;
(Siii) L is a local Q_0^E-martingale null at 0 such that $\mathscr{E}(L)$ is a positive Q_0^E-martingale and S is a $Q(L)$-sigma-martingale, where $Q(L)$ is defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathscr{E}(L)_T$.

We call (Γ, ψ, L) a *special solution* of (51), (52) if furthermore

(Siv) $\int \psi \,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^{e,f}$;
(Sv) $E_P[\mathscr{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P} \log(\mathscr{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P})] < \infty$, i.e., the probability measure $Q(L)$ defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathscr{E}(L)_T$ has finite relative entropy with respect to P.

If S is locally bounded, we say that (Γ, ψ, L) is an *orthogonal solution* of (51), (52) if it satisfies (51), (52), *(Si)*, *(Sii)* and

(Siii′) L is a local Q_0^E-martingale null at 0 and strongly Q_0^E-orthogonal to every component of S and such that $\mathscr{E}(L)$ is positive.

Under the assumption that S is locally bounded,

$$\text{a triple } (\Gamma, \psi, L) \text{ is a solution of (51), (52) if and only if}$$
$$\text{it is an orthogonal solution and } \mathscr{E}(L) \text{ is a } Q_0^E\text{-martingale.} \tag{53}$$

To see this, note first that a locally bounded S is a $Q(L)$-sigma-martingale if and only if $\mathscr{E}(L)S$ is a local Q_0^E-martingale, under the assumption that $Q(L)$ is a probability measure. If (Γ, ψ, L) is a solution, then (Siii) holds and all of $\mathscr{E}(L)S$, $\mathscr{E}(L)$ and S are local Q_0^E-martingales. Hence $\mathscr{E}(L)$ is strongly Q_0^E-orthogonal to every component of S, and therefore so is L. Conversely, if (Siii′) holds, then $\mathscr{E}(L)$ is like L strongly Q_0^E-orthogonal to every component of the local Q_0^E-martingale S. Hence $\mathscr{E}(L)S$ is a local Q_0^E-martingale and thus S is a $Q(L)$-sigma-martingale if $\mathscr{E}(L)$ is a Q_0^E-martingale.

Our main result in this section is then

Theorem 5 *Assume that H satisfies* (13). *Then the indifference value process h is the first component of the unique special solution of the BSDE* (51), (52).

Theorem 5 looks at first glance like Theorem 13 of Mania and Schweizer [19]. The important difference, however, is that we do not suppose that the filtration $\mathbb{F}$ is continuous, i.e., that all local P-martingales are continuous. If $\mathbb{F}$ is continuous, then $\frac{1}{\gamma} \log \mathscr{E}(L) = L/\gamma - \frac{\gamma}{2}\langle L/\gamma\rangle$ and Theorem 5 corresponds to Theorem 13 of Mania and Schweizer [19]. (Since H is allowed to be unbounded in Theorem 5, there are some differences in the integrability properties.) However, recovering the latter result in precise form and almost full strength from Theorem 5 requires some additional work which we discuss at the end of this section. The derivation in [19] uses the martingale optimality principle, the existence of an optimal strategy for the indifference value process, and a comparison theorem for BSDEs. Our proof is completely different; it is based on our results for the $FER^\star(H)$ and its relation to the indifference value.

Theorem 4.4 of Becherer [2] is another similar result. Instead of a continuous filtration, the framework in [2] has a continuous price process driven by Brownian motions, and a filtration generated by these and a random measure allowing the modeling of non-predictable events. Again, to regain from Theorem 5 the same statement as in Theorem 4.4 of Becherer [2], some additional work is necessary.

In Corollary 3.6 of the earlier paper [1], Becherer gives a characterization of $\frac{\mathrm{d}Q_H^E}{\mathrm{d}Q_0^E}$ in a locally bounded semimartingale model. Theorem 5 can be viewed as a dynamic extension of that result to a general semimartingale model.

Proof of Theorem 5 By Lemma 1, (13) implies that $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f} \neq \emptyset$, and so Theorem 3 and (9) yield

$$h_t = k_t^H - k_t^0 = h_0 + \frac{1}{\gamma}\log\frac{\mathscr{E}(N^H)_t}{\mathscr{E}(N^0)_t} + \int_0^t (\eta_s^H - \eta_s^0)\mathrm{d}S_s, \quad 0 \le t \le T,$$

where (N^H, η^H, k_0^H) and (N^0, η^0, k_0^0) are the $FER^\star(H)$ and $FER^\star(0)$; see Proposition 2 for their properties. Then $\psi := \eta^H - \eta^0$ is in $L(S)$ and $\int \psi \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$. By Bayes' formula, $\mathscr{E}(N^H)/\mathscr{E}(N^0)$ is the Q_0^E-density process of Q_H^E, and so it is a positive Q_0^E-martingale and its stochastic logarithm L, defined by $\mathscr{E}(L) = \mathscr{E}(N^H)/\mathscr{E}(N^0)$, is a local Q_0^E-martingale null at 0. Moreover, $\frac{\mathrm{d}Q(L)}{\mathrm{d}P} = \mathscr{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P} = \frac{\mathrm{d}Q_H^E}{\mathrm{d}P}$ shows $Q(L) = Q_H^E$. Hence S is a $Q(L)$-sigma-martingale and (Sv) is satisfied because Q_H^E has finite relative entropy with respect to P. Since $h_T = H$ by definition, we see that h is the first component of a special solution of the BSDE (51), (52).

To prove uniqueness, let (Γ, ψ, L) be any special solution of (51), (52). Denote by (N^0, η^0, k_0^0) the unique $FER^\star(0)$, and define

$$N := N^0 + L + [N^0, L], \qquad \eta := \eta^0 + \psi \quad \text{and} \quad k_0 := k_0^0 + \Gamma_0. \tag{54}$$

We claim that

$$(N, \eta, k_0) \quad \text{is the unique } FER^\star(H). \tag{55}$$

For the proof, we first note that $\mathscr{E}(N^0)\mathscr{E}(L) = \mathscr{E}(N^0 + L + [N^0, L]) = \mathscr{E}(N)$ by Yor's formula. Using (51), (52) and (7) for $H = 0$ thus yields

$$\begin{aligned} H &= \frac{1}{\gamma}\log(\mathscr{E}(N^0)_T\mathscr{E}(L)_T) + \int_0^T (\eta_s^0 + \psi_s)\mathrm{d}S_s + k_0^0 + \Gamma_0 \\ &= \frac{1}{\gamma}\log\mathscr{E}(N)_T + \int_0^T \eta_s \mathrm{d}S_s + k_0. \end{aligned}$$

Therefore (N, η, k_0) satisfies (7) for H, and it is enough to show that the assumptions on N and η for $FER^\star(H)$ are fulfilled. By Bayes' formula, $\mathscr{E}(N) = \mathscr{E}(N^0)\mathscr{E}(L)$ is a positive P-martingale, because $\mathscr{E}(L)$ is a positive Q_0^E-martingale by (Siii) and $\mathscr{E}(N^0)$ is the P-density process of Q_0^E. Writing next

$$\frac{\mathrm{d}P(N)}{\mathrm{d}Q_0^E} = \frac{\mathrm{d}P(N)}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}Q_0^E} = \mathscr{E}(N)_T/\mathscr{E}(N^0)_T = \mathscr{E}(L)_T,$$

we see that $P(N) = Q(L)$ which implies that

$$I(P(N)|P) = E_P\left[\mathscr{E}(L)_T\frac{\mathrm{d}Q_0^E}{\mathrm{d}P}\log\left(\mathscr{E}(L)_T\frac{\mathrm{d}Q_0^E}{\mathrm{d}P}\right)\right] < \infty$$

by (Sv) and that S is a $P(N)$-sigma-martingale by (Siii). Because (N^0, η^0, k_0^0) is the $FER^\star(0)$, $\int \eta \mathrm{d}S = \int \eta^0 \mathrm{d}S + \int \psi \mathrm{d}S$ is by Proposition 2 and (Siv) a Q-martingale for every $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$, hence also for $P(N)$ and Q_H^E, and so (N, η, k_0) is an $FER(H)$ satisfying (c) from Proposition 2. This implies (55). Uniqueness of the $FER^\star(H)$ and (54) now imply that Γ_0, ψ are unique; so is L due to $\mathscr{E}(L) = \mathscr{E}(N)/\mathscr{E}(N^0)$, and finally also Γ by (51). This ends the proof. □

The above argument shows in particular a close link between the $FER^\star(H)$ and the BSDE (51), (52). Provided we have the $FER^\star(0)$, we can construct $FER^\star(H)$ from the special solution of (51), (52), and vice versa. This is familiar from exponential utility indifference valuation; indeed, knowing $FER^\star(0)$ corresponds to knowing the minimal 0-entropy measure Q_0^E.

Remark 4 If S is locally bounded and H is bounded, there is another way to prove uniqueness of the first component of a special solution of the BSDE (51), (52), which we briefly sketch here. If (Γ, ψ, L) is a special solution of (51), (52), the idea is to show that Γ equals the indifference value process h, which then yields the desired uniqueness result. Let $t \in [0, T]$ and replace in the definition of $\mathscr{A}_t^H$ the condition that $\int \vartheta \mathrm{d}S$ is a Q-supermartingale for every $Q \in \mathbb{P}_H^{e,f}$ by assuming that it is a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$. We do the analogous change for $\mathscr{A}_t^0$ and note that this does not affect the values of V_t^H and V_t^0, as mentioned after the proof of Theorem 3. We now apply Proposition 3 of Mania and Schweizer [19] to obtain

$$h_t = \frac{1}{\gamma}\log \operatorname*{ess\,inf}_{\vartheta \in \mathscr{A}_t^H} E_{Q_0^E}\left[\exp\left(\gamma H - \gamma\int_t^T \vartheta_s \mathrm{d}S_s\right)\middle|\mathscr{F}_t\right]. \tag{56}$$

Using (51), (52) gives

$$\gamma H = \gamma\Gamma_0 + \log\mathscr{E}(L)_T + \gamma\int_0^T \psi_s \mathrm{d}S_s = \gamma\Gamma_t + \log\frac{\mathscr{E}(L)_T}{\mathscr{E}(L)_t} + \gamma\int_t^T \psi_s \mathrm{d}S_s,$$

which we plug into (56) to obtain

$$h_t = \Gamma_t + \frac{1}{\gamma} \log \operatorname*{ess\,inf}_{\vartheta \in \mathscr{A}_t^H} E_{Q(L)}\left[\exp\left(\gamma \int_t^T (\psi_s - \vartheta_s) \mathrm{d}S_s\right) \middle| \mathscr{F}_t\right] =: \Gamma_t + \frac{1}{\gamma} \log \Lambda,$$

where the probability measure $Q(L)$ is defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathscr{E}(L)_T$. To show that $\Lambda = 1$, we first note that $Q(L) \in \mathbb{P}_0^{e,f}$ by (Sv), $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ by Lemma 1, and $\int \psi \mathrm{d}S$ as well as $\int \vartheta \mathrm{d}S$ are Q-martingales for every $Q \in \mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ by (Siv) and because $\vartheta \in \mathscr{A}_t^H$. Jensen's inequality then yields $\Lambda \geq 1$, and we obtain $\Lambda \leq 1$ by the choice $\vartheta^\star := \psi \in \mathscr{A}_t^H$. Note that also for this uniqueness proof, we have used the assumption that (Γ, ψ, L) is a *special* solution of the BSDE (51), (52), i.e., that it also satisfies (Siv), (Sv).

We have seen in Sect. 3 that the difference between $FER(H)$ and the (unique) $FER^\star(H)$ is an issue of integrability. The same thing happens here: The next example shows that the BSDE (51), (52) may have many solutions if we omit the requirement (Siv) (which corresponds to (d) in Proposition 2).

Example 2 As in Example 1, take independent P-Brownian motions W and $W^\perp$, their P-augmented filtration $\mathbb{F}$ and $d = 1$, $S = W$, $H \equiv 0$. Then $Q_0^E = P$ and $(0, 0, 0)$ is the unique special solution of (51), (52).

As in Example 1, take $N^0 = W^\perp$ and use Proposition 1 of Emery et al. [8] to find for any $c \in \mathbb{R}$ a process $\psi(c)$ in $L(S)$ such that

$$-\frac{1}{\gamma} \log \mathscr{E}\left(N^0\right)_T - c = \int_0^T \psi_s(c) \mathrm{d}S_s \quad P\text{-a.s.}$$

If we then set $\Gamma_t(c) := c + \frac{1}{\gamma} \log \mathscr{E}(N^0)_t + \int_0^t \psi_s(c) \mathrm{d}S_s$ for $t \in [0, T]$, we easily see as in Example 1 that $(\Gamma(c), \psi(c), N^0)$ is a solution to (51), (52) and satisfies (Sv), but not (Siv). So we clearly have multiple solutions.

Theorem 5 allows us to obtain a result similar to Proposition 3.

Corollary 2 *Assume that H satisfies* (13). *Then we have for any probability measure $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$ and $t \in [0, T]$ that*

$$h_t = E_Q[H | \mathscr{F}_t] - \frac{1}{\gamma} E_Q[\log \mathscr{E}(L)_{t,T} | \mathscr{F}_t], \tag{57}$$

where L is the third component of the unique special solution of the BSDE (51), (52). *In particular,*

$$h_0 = E_{Q_0^E}[H] + \frac{1}{\gamma} I(Q_0^E | Q(L)), \tag{58}$$

where $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathscr{E}(L)_T$.

Proof Equation (57) follows from Theorem 5 by taking conditional Q-expectations between t and T in (51), using (52) and (Siv). Equation (58) follows for $Q = Q_0^E$. □

Remark 5 Corollary 2 raises the question if one can find a probability measure $Q \in \mathbb{P}_0^{e,f}$ such that the indifference value is the Q-conditional expectation of H. From (57) we see that $\log \mathscr{E}(L)$ must then be a Q-martingale, and if we write the Q_0^E-density process of Q as $\mathscr{E}(R)$ for some local Q_0^E-martingale R, Bayes' formula tells us that we want $\mathscr{E}(R)\log \mathscr{E}(L)$ to be a Q_0^E-martingale. Itô's formula gives

$$\begin{aligned} \mathrm{d}(\mathscr{E}(R)\log\mathscr{E}(L))_t &= \log\mathscr{E}(L)_{t-}\mathrm{d}\mathscr{E}(R)_t + \frac{\mathscr{E}(R)_{t-}}{\mathscr{E}(L)_{t-}}\mathrm{d}\mathscr{E}(L)_t \\ &\quad + \mathscr{E}(R)_{t-}\mathrm{d}\left[L^c, R^c - \frac{1}{2}L^c\right]_t \\ &\quad + \mathscr{E}(R)_{t-}((\Delta R_t + 1)\log(1+\Delta L_t) - \Delta L_t), \end{aligned}$$

where L^c and R^c denote the continuous local Q_0^E-martingale parts of L and R. For $\mathscr{E}(R)\log\mathscr{E}(L)$ to be a local Q_0^E-martingale, we must have that $R^c = \frac{1}{2}L^c$ on $\{L^c \neq 0\}$ and $\Delta R_t = \frac{\Delta L_t - \log(1+\Delta L_t)}{\log(1+\Delta L_t)}$ on $\{\Delta L_t \neq 0\}$. Therefore, we define $R = R^c + R^d$ by

$$R_t^c := \frac{1}{2}L_t^c \quad \text{and} \quad R_t^d := \sum_{0<s\le t} \frac{\Delta L_s - \log(1+\Delta L_s)}{\log(1+\Delta L_s)} I_{\Delta L_s \neq 0} - A_t, \tag{59}$$

where A is the dual predictable projection under Q_0^E of the sum in (59). Note that R^d is well defined, since $\Delta L_s > -1$, $\Delta L_s \neq 0$ implies that

$$\left|\frac{\Delta L_s - \log(1+\Delta L_s)}{\log(1+\Delta L_s)}\right| \le |\Delta L_s|;$$

in fact, $\log(1+x) \ge \frac{x}{1+x}$ for $x > -1$ implies that $|\frac{x-\log(1+x)}{\log(1+x)}| \le |x|$ for $x > -1$, $x \neq 0$. By this construction, $\mathscr{E}(R)$ and $\mathscr{E}(R)\log\mathscr{E}(L)$ are local Q_0^E-martingales, but it is not clear whether they are true Q_0^E-martingales. If they are and if Q defined by $\frac{\mathrm{d}Q}{\mathrm{d}Q_0^E} := \mathscr{E}(R)_T$ is in $\mathbb{P}_0^{e,f}$, then we obtain indeed $h_t = E_Q[H|\mathscr{F}_t]$ for all $t \in [0,T]$. In general, this representation is not linear in H since the probability measure Q may (via L) depend on H. Mania and Schweizer [19] showed in their Proposition 11 that a representation of this type exists if the filtration is continuous and H is bounded, in which case $R = \frac{1}{2}L$.

Becherer [2] and Mania and Schweizer [19] show *BMO*-estimates for all components of the solution to the BSDE for the indifference value process h. It seems doubtful if one can obtain such results in our general framework here, but under a mild additional assumption, we can still characterize (Siv) via *BMO*-properties without being more specific about the filtration $\mathbb{F}$; see Theorem 6 below.

The indifference hedging strategy β is defined as the difference of the strategies which attain $V_0^H(h_0)$ and $V_0^0(0)$, i.e., as that extra trading we do in the optimization which can be attributed to the presence of a claim. If H satisfies (13), we have $\beta = \eta^H - \eta^0 = \psi$ by (32) and the proof of Theorem 5, where ψ is the second component of the unique special solution of the BSDE (51), (52). Hence it is of particular interest to know when $\int \psi \, \mathrm{d}S$ is a $BMO(Q_0^E)$-martingale.

Theorem 6 *Assume that S is continuous, H is bounded and there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L \log L}(P)$. Let (Γ, ψ, L) be a solution of the BSDE* (51), (52) *which satisfies* (Sv). *Then the following are equivalent*:

(a) (Γ, ψ, L) *is the special solution of* (51), (52), *i.e., it also satisfies* (Siv);
(b) L *is a $BMO(Q_0^E)$-martingale, $\mathscr{E}(L)$ satisfies condition (J), and $\int \psi \, \mathrm{d}S$ is a Q_0^E-martingale*;
(c) $\int \psi \, \mathrm{d}S$ *is a $BMO(Q_0^E)$-martingale*;
(d) $\int \psi \, \mathrm{d}M$ *is a $BMO(P)$-martingale, where M is the P-local martingale part of S*;
(e) *there exists $\varepsilon > 0$ such that $E_P[\exp(\varepsilon[\int \psi \, \mathrm{d}S]_T)] < \infty$.*

Proof "(a) $\Longrightarrow$ (b)". Denote by (N^H, η^H, k_0^H) and (N^0, η^0, k_0^0) the unique $FER^\star(H)$ and $FER^\star(0)$. Theorem 2 implies that N^H, N^0 are $BMO(P)$-martingales and $\mathscr{E}(N^H)$, $\mathscr{E}(N^0)$ satisfy condition (J), say with constants C^H and C^0. By the proof of Theorem 5, we have $\mathscr{E}(L) = \mathscr{E}(N^H)/\mathscr{E}(N^0)$ and thus $\mathscr{E}(L)$ satisfies condition (J) with constant $C^H C^0$. Since $1/\mathscr{E}(N^0)$ is the Q_0^E-density process of P, $\mathscr{E}(N^0)^{-1} = \mathscr{E}(\widehat{N}^0)$ for a local Q_0^E-martingale $\widehat{N}^0$, and so $\mathscr{E}(L) = \mathscr{E}(N^H + \widehat{N}^0 + [N^H, \widehat{N}^0])$ by Yor's formula. Due to the properties of N^0 and N^H, both $\widehat{N}^0$ and $N^H + [N^H, \widehat{N}^0]$ are $BMO(Q_0^E)$-martingales by Propositions 6 and 7 of Doléans-Dade and Meyer [7], and hence so is $L = \widehat{N}^0 + N^H + [N^H, \widehat{N}^0]$. Finally, $\int \psi \, \mathrm{d}S$ is a Q_0^E-martingale by Siv).

"(b) $\Longrightarrow$ (c)", "(c) $\Longrightarrow$ (d)" and "(d) $\Longrightarrow$ (e)". These go along the same lines as the proofs of the corresponding implications in Theorem 2. Instead of (7) we take (51), (52), and we replace $P(N^H)$ by Q_0^E.

"(e) $\Longrightarrow$ (a)". Like for the corresponding implication in Theorem 2, we obtain that $\int \psi \, \mathrm{d}S$ is a square-integrable Q-martingale for any $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$, which implies (Siv). □

Remark 6 Example 2 also shows that even if the assumptions of Theorem 6 are satisfied, none of the equivalent statements (a)–(e) need hold. This is another way of saying that there exist solutions of (51), (52) which are not special solutions.

Corollary 3 *Suppose the assumptions of Theorem 6 hold. Let (Γ, ψ, L) be an orthogonal solution of the BSDE* (51), (52). *Then (Γ, ψ, L) is the special solution of* (51), (52) *if and only if both L and $\int \psi \, \mathrm{d}S$ are $BMO(Q_0^E)$-martingales and $\mathscr{E}(L)$ is a Q_0^E-martingale which satisfies condition* (J).

Proof The "only if" part follows immediately from Theorem 6. For the "if" part, note first that (Γ, ψ, L) is a solution of (51), (52) by (53). So we need only show that (Γ, ψ, L) satisfies (Sv) in view of Theorem 6. We first prove that $\int \psi \mathrm{d}S$ is a $BMO(Q(L))$-martingale, where $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} = \mathscr{E}(L)_T$. Because $1/\mathscr{E}(L)$ is the $Q(L)$-density process of Q_0^E, it can be written as $\mathscr{E}(L)^{-1} = \mathscr{E}(\widehat{L})$ for a local $Q(L)$-martingale $\widehat{L}$ which must satisfy $L + \widehat{L} + [L, \widehat{L}] = 0$ by Yor's formula. The continuity of S and the strong Q_0^E-orthogonality of L to S entail

$$\left[\int \psi \mathrm{d}S, \widehat{L}\right] = -\left[\int \psi \mathrm{d}S, L\right] = 0.$$

This yields by Proposition 7 of Doléans-Dade and Meyer [7] that $\int \psi \mathrm{d}S$ is a $BMO(Q(L))$-martingale. For the second component η^0 of the $FER^\star(0)$, we similarly have that $\int \eta^0 \mathrm{d}S$ is a $BMO(Q(L))$-martingale since $\int \eta^0 \mathrm{d}S$ is a $BMO(Q_0^E)$-martingale by Theorem 2. Because (Γ, ψ, L) is a solution of (51), (52), we can write

$$\log \mathscr{E}(L)_T = -\gamma \int_0^T \psi_s \mathrm{d}S_s + \gamma H - \gamma \Gamma_0,$$

and similarly, we have for the $FER^\star(0)$ (N^0, η^0, k_0^0) that

$$\log \frac{\mathrm{d}Q_0^E}{\mathrm{d}P} = \log \mathscr{E}(N^0)_T = -\gamma \int_0^T \eta_s^0 \mathrm{d}S_s - \gamma k_0^0.$$

Because $\int (\eta^0 + \psi)\mathrm{d}S$ is a $BMO(Q(L))$-martingale, we thus obtain

$$\begin{aligned} E_{Q(L)}\left[\log\left(\mathscr{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P}\right)\right] &= -\gamma\Gamma_0 - \gamma k_0^0 + \gamma E_{Q(L)}\left[H - \int_0^T (\eta_s^0 + \psi_s)\mathrm{d}S_s\right] \\ &= -\gamma\Gamma_0 - \gamma k_0^0 + \gamma E_{Q(L)}[H] < \infty \end{aligned}$$

since H is bounded. Hence (Γ, ψ, L) satisfies (Sv) and we are done. □

Corollary 3 allows us to recover Theorem 13 of Mania and Schweizer [19] from our Theorem 5. However, this still requires some work which is done in the next two results. A similar approach can be used to recover Theorem 4.4 of Becherer [2] from our Theorem 5, but we do not detail this here. Although the following lemma is a special case of Proposition 7 of Mania and Schweizer [19], we give the proof here as well, both for completeness and because it is quite simple in this case.

Lemma 2 *Assume that the filtration $\mathbb{F}$ is continuous, H is bounded and let (Γ, ψ, L) be an orthogonal solution of the BSDE (51), (52) with bounded first component Γ. Then L and $\int \psi \mathrm{d}S$ are $BMO(Q_0^E)$-martingales.*

Proof If L and $\int \psi \mathrm{d}S$ are true Q_0^E-martingales, (51) yields by continuity of L

$$E_{Q_0^E}[\langle L\rangle_T - \langle L\rangle_\tau | \mathscr{F}_\tau] = 2\gamma E_{Q_0^E}[\Gamma_\tau - \Gamma_T | \mathscr{F}_\tau] \quad \text{for any stopping time } \tau. \tag{60}$$

Because Γ is bounded, the right-hand side of (60) is bounded independently of τ, and thus L is a $BMO(Q_0^E)$-martingale. This implies that $(E_{Q_0^E}[\langle L\rangle_T|\mathscr{F}_s])_{0\le s\le T}$ is also a continuous $BMO(Q_0^E)$-martingale, because

$$E_{Q_0^E}[|\langle L\rangle_T - E_{Q_0^E}[\langle L\rangle_T|\mathscr{F}_\tau]||\mathscr{F}_\tau] \le 2E_{Q_0^E}[\langle L\rangle_T - \langle L\rangle_\tau|\mathscr{F}_\tau] \le 2\|L\|^2_{BMO_2(Q_0^E)}$$

for any stopping time τ. Taking conditional Q_0^E-expectations in (51) with $t = T$ gives

$$\int_0^s \psi_y \mathrm{d}S_y = E_{Q_0^E}[\Gamma_T - \Gamma_0|\mathscr{F}_s] - \frac{1}{\gamma}L_s + \frac{1}{2\gamma}E_{Q_0^E}[\langle L\rangle_T|\mathscr{F}_s], \quad 0 \le s \le T,$$

and so $\int \psi \mathrm{d}S$ is a $BMO(Q_0^E)$-martingale as well. Note that we obtain bounds for the $BMO_2(Q_0^E)$-norms of L and $\int \psi \mathrm{d}S$ that depend on Γ (and γ) alone.

For general L and $\int \psi \mathrm{d}S$, we stop at τ_n and apply the above argument with T replaced by τ_n. Letting $n \to \infty$ then completes the proof. □

A closer look at the proof of Lemma 2 shows that we did not use the property that L is strongly Q_0^E-orthogonal to S. However, this is of course necessary if we want to prove a uniqueness result. By combining Lemma 2 and Corollary 3, we obtain the following sufficient conditions for the uniqueness of an orthogonal solution of (51), (52) with bounded first component.

Proposition 4 *Assume that $\mathbb{F}$ is continuous, H is bounded, and there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. Then the indifference value process h is the first component of the unique orthogonal solution of* (51), (52) *with bounded first component. Moreover, L and $\int \psi \mathrm{d}S$ are $BMO(Q_0^E)$-martingales.*

Proof By Theorem 5 and (53), h is the first component of an orthogonal solution of (51), (52). Using the definition (3) of h and $V_t^H(h_t) = \exp(-\gamma h_t)V_t^H(0)$ easily implies that the indifference value process h is bounded by $\|H\|_{L^\infty(P)}$. If (Γ, ψ, L) is any orthogonal solution of the BSDE (51), (52) with bounded Γ, then L and $\int \psi \mathrm{d}S$ are $BMO(Q_0^E)$-martingales by Lemma 2. By Corollary 3, (Γ, ψ, L) is then a special solution, which is unique by Theorem 5. □

Proposition 4 is almost identical to Theorem 13 in Mania and Schweizer [19]; the only difference is that we have here the additional assumption that there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. The explanation for this is that we actually prove more than we really need for Proposition 4. Mania and Schweizer [19] use a comparison result for BSDEs (their Theorem 8) to deduce directly that one has uniqueness of orthogonal solutions to the BSDE within the class of those with bounded first component. In contrast, the proof of Proposition 4 actually shows that under the $R_{L\log L}$-condition, any solution with bounded first component is even a special solution—and then one appeals to Theorem 5 which asserts uniqueness within that class.

6 Application to a Brownian Setting

In this section, we consider as a special case a model with one risky asset driven by a Brownian motion and a claim coming from a second, correlated Brownian motion. All processes are indexed by $0 \leq s \leq T$. Let W and Y be two Brownian motions with constant instantaneous correlation ϱ satisfying $|\rho| < 1$. Choose as $\mathbb{F}$ the P-augmentation of the filtration generated by the pair (W, Y), and denote by $\mathbb{Y} = (\mathscr{Y}_s)_{0 \leq s \leq T}$ the P-augmentation of the filtration generated by Y alone.

As usual, the risk-free *bank account* has zero interest rate. The single *tradable stock* has a price process given by

$$\mathrm{d}S_s = \mu_s S_s \mathrm{d}s + \sigma_s S_s \mathrm{d}W_s, \quad 0 \leq s \leq T, \; S_0 > 0, \tag{61}$$

where drift μ and volatility σ are $\mathbb{F}$-predictable processes. We assume for simplicity that μ is bounded and σ is bounded away from zero and infinity. We further assume that

the instantaneous Sharpe ratio $\frac{\mu}{\sigma}$ of the tradable stock is $\mathbb{Y}$-predictable.

In the notation of Sect. 2, $S = S_0 + M + \int \lambda \mathrm{d}\langle M \rangle$, where $M := \int \sigma S \mathrm{d}W$ is a local $(\mathbb{F}, P)$-martingale and $\lambda := \frac{\mu}{\sigma} \frac{1}{\sigma S}$ is $\mathbb{F}$-predictable. Since μ is bounded and σ is bounded away from zero, the Sharpe ratio $\frac{\mu}{\sigma}$ is also bounded, and thus $\int \lambda \mathrm{d}M = \int \frac{\mu}{\sigma} \mathrm{d}W$ is a $BMO(\mathbb{F}, P)$-martingale and $\mathscr{E}(-\int \lambda \mathrm{d}M)$ is an $(\mathbb{F}, P)$-martingale. We suppose that the contingent claim H is a bounded $\mathscr{Y}_T$-measurable random variable. Together with the structure of S in (61), this assumption on H formalizes the idea that the payoff H is driven by Y, whereas hedging can only be done in S which is imperfectly correlated with the factor Y.

In the literature, there are three main approaches to obtain explicit formulas for the resulting optimization problem (2). In a Markovian setting, Henderson [13], Henderson and Hobson [14, 15], and Musiela and Zariphopoulou [20], among others, first derive the Hamilton-Jacobi-Bellman nonlinear PDE for the value function of the underlying stochastic control problem. This PDE is then linearized by a power transformation with a constant exponent, called the *distortion power*, which corresponds to δ_0^H from Theorem 4 and Corollary 1. This method works only if one has a Markovian model. Using general techniques, Tehranchi [25] first proves a Hölder-type inequality, which he then applies to the portfolio optimization problem. The distortion power there arises as an exponent in the Hölder-type inequality. A third approach based on martingale arguments allows us in [10] to consider a more general framework with a fairly general stochastic correlation ϱ. In [10], we prove that the explicit form of the indifference value from Musiela and Zariphopoulou [20] or Tehranchi [25] is preserved, except that the distortion power, which is shown to exist but not explicitly determined, may be random and depend on H like in our general semimartingale model; compare Theorem 4 and Corollary 1.

We give here another proof based on the results of the previous sections. While there are no new results, the arguments in comparison to [10] are easier and shorter, give new insights, and show the advantage of $FER^\star(H)$ compared to the BSDE formulation (51), (52) in Sect. 5. Indeed, $FER^\star(H)$ is a representation under the original probability measure P, whereas in the BSDE formulation (51), (52), one must first determine the minimal 0-entropy measure.

Proposition 5 *For any $t \in [0, T]$ and any $\mathscr{F}_t$-measurable random variable x_t,*

$$V_t^H(x_t) = -\exp(-\gamma x_t) E_{\widehat{P}}[|\Psi_t^H|^{1-|\varrho|^2} | \mathscr{Y}_t]^{\frac{1}{1-|\varrho|^2}},$$

where $\Psi_t^H = \exp(\gamma H - \frac{1}{2}\int_t^T |\frac{\mu_s}{\sigma_s}|^2 \mathrm{d}s)$ and the minimal martingale measure $\widehat{P}$ is given by

$$\frac{\mathrm{d}\widehat{P}}{\mathrm{d}P} = \mathscr{E}\Big(-\int \frac{\mu}{\sigma}\mathrm{d}W\Big)_T. \tag{62}$$

The exponential utility indifference value h_t of H at time t equals

$$h_t = \frac{1}{\gamma(1-|\varrho|^2)} \log \frac{E_{\widehat{P}}[|\Psi_t^H|^{1-|\varrho|^2} | \mathscr{Y}_t]}{E_{\widehat{P}}[|\Psi_t^0|^{1-|\varrho|^2} | \mathscr{Y}_t]}.$$

In Corollary 1, we have shown that

$$\begin{aligned} h_t(\omega) = {} & \frac{1}{\gamma}\log(E_{\hat{P}}[|\Psi_t^H|^{1/\delta} | \mathscr{F}_t](\omega))^{\delta}|_{\delta=\delta_t^H(\omega)} \\ & - \frac{1}{\gamma}\log(E_{\widehat{P}}[|\Psi_t^0|^{1/\delta'} | \mathscr{F}_t](\omega))^{\delta'}|_{\delta'=\delta_t^0(\omega)}, \end{aligned}$$

and have related $1/\delta^H$ to a kind of distance of H from attainability. Here we have $1/\delta^H = 1 - |\varrho|^2$, which confirms our interpretation: The closer $1/\delta^H$ is to one, the greater is the distance of H from being attainable, because a smaller correlation ϱ between W and Y makes hedging more difficult.

Proof of Proposition 5 The idea is to explicitly derive the $FER^\star(H)$ and $FER^\star(0)$, from which the result follows by Theorem 3. In view of Proposition 1 and (10), we thus look for suitable real-valued processes $\widetilde{N}^H$ and $\widetilde{\eta}^H$ and an $\mathscr{F}_t$-measurable random variable k_t^H such that

$$H = \frac{1}{\gamma}\log \mathscr{E}(\widetilde{N}^H)_{t,T} + \int_t^T \widetilde{\eta}_s^H \sigma_s S_s \mathrm{d}\widehat{W}_s + \frac{1}{2\gamma}\int_t^T \left|\frac{\mu_s}{\sigma_s}\right|^2 \mathrm{d}s + k_t^H, \tag{63}$$

where $\widehat{W} := W + \int \frac{\mu}{\sigma}\mathrm{d}s$ is by Girsanov's theorem a Brownian motion under the minimal martingale measure $\hat{P}$ given by (62). Using Itô's representation theorem as in Lemma 1.6.7 of Karatzas and Shreve [17] for $|\Psi_t^H|^{1-|\varrho|^2}$ under $\mathbb{Y}$ and $\widehat{P}$ restricted to $\mathscr{Y}_T$, we can find a $\mathbb{Y}$-predictable process ζ with $E_{\hat{P}}[\int_0^T |\zeta_s|^2 \mathrm{d}s] < \infty$ such that

$$|\Psi_t^H|^{1-|\varrho|^2} = E_{\widehat{P}}[|\Psi_t^H|^{1-|\varrho|^2} | \mathscr{Y}_t] \mathscr{E}\Big(\int \zeta \,\mathrm{d}\widehat{Y}\Big)_{t,T}, \tag{64}$$

where the $(\mathbb{Y}, \widehat{P})$-Brownian motion $\widehat{Y}$ is defined by

$$\widehat{Y}_s := Y_s + \int_0^s \varrho \frac{\mu_y}{\sigma_y} \mathrm{d}y \quad \text{for } s \in [0, T].$$

Note that this argument uses that Ψ_t^H is $\mathscr{Y}_T$-measurable because $\frac{\mu}{\sigma}$ is $\mathbb{Y}$-predictable and H is $\mathscr{Y}_T$-measurable by assumption. We can write $\widehat{Y} = \varrho \widehat{W} + \sqrt{1 - |\varrho|^2}\widehat{W}^\perp$ for an $(\mathbb{F}, \hat{P})$-Brownian motion $\widehat{W}^\perp$ independent of $\widehat{W}$. Taking the logarithm in (64) results in

$$H = \frac{1}{\gamma} \int_t^T \frac{\zeta_s}{1 - |\varrho|^2} \mathrm{d}\widehat{Y}_s - \frac{1}{2\gamma} \int_t^T \frac{|\zeta_s|^2}{1 - |\varrho|^2} \mathrm{d}s + \frac{1}{2\gamma} \int_t^T \left| \frac{\mu_s}{\sigma_s} \right|^2 \mathrm{d}s + k_t^H,$$

where

$$k_t^H := \frac{1}{\gamma(1 - |\varrho|^2)} \log E_{\widehat{P}}[|\Psi_t^H|^{1-|\varrho|^2} | \mathscr{Y}_t].$$

But this is (63) with

$$\widetilde{N}^H := \int \frac{\zeta}{\sqrt{1 - |\varrho|^2}} \mathrm{d}\widehat{W}^\perp \quad \text{and} \quad \widetilde{\eta}^H := \frac{\varrho \zeta}{\gamma(1 - |\varrho|^2)} \frac{1}{\sigma S}.$$

Clearly, $\widetilde{N}^H$ is a local $\widehat{P}$-martingale strongly $\widehat{P}$-orthogonal to S, hence also a local P-martingale strongly P-orthogonal to M. Moreover, Ψ_t^H is bounded away from zero and infinity, which implies by (64) that $\mathscr{E}(\int \zeta \mathrm{d}\widehat{Y})$ is uniformly bounded away from zero and infinity. By Theorem 3.4 of Kazamaki [18], $\int \zeta \mathrm{d}\widehat{Y}$ is then a $BMO(\mathbb{F}, \widehat{P})$-martingale and thus so is $\widetilde{N}^H$ because

$$\langle \widetilde{N}^H \rangle = \frac{1}{1 - |\varrho|^2} \int |\zeta|^2 \mathrm{d}s = \frac{1}{1 - |\varrho|^2} \left\langle \int \zeta \mathrm{d}\widehat{Y} \right\rangle.$$

This implies first that $\mathscr{E}(\widetilde{N}^H)$ is an $(\mathbb{F}, \widehat{P})$-martingale so that $\mathscr{E}(\widetilde{N}^H)\mathscr{E}(-\int \lambda \mathrm{d}M)$ is an $(\mathbb{F}, P)$-martingale, and then that also

$$\begin{aligned} \int (\gamma \widetilde{\eta}^H + \lambda) \mathrm{d}S &= \int \gamma \widetilde{\eta}^H \sigma S \mathrm{d}\widehat{W} + \int \frac{\mu}{\sigma} \mathrm{d}\widehat{W} \\ &= \frac{1}{1 - |\varrho|^2} \int \zeta \mathrm{d}\widehat{Y} - \widetilde{N}^H + \int \frac{\mu}{\sigma} \mathrm{d}\widehat{W} \end{aligned}$$

is a $BMO(\mathbb{F}, \widehat{P})$-martingale. So if we set $\frac{\mathrm{d}P(N^H)}{\mathrm{d}\widehat{P}} = \mathscr{E}(\widetilde{N}^H)_T$, then $\int (\widetilde{\eta}^H + \frac{1}{\gamma}\lambda) \mathrm{d}S$ is also a $BMO(\mathbb{F}, P(N^H))$-martingale by Theorem 3.6 of Kazamaki [18]. By Proposition 1, $(\widetilde{N}^H - \int \frac{\mu}{\sigma} \mathrm{d}W, \widetilde{\eta}^H + \frac{\mu}{\gamma\sigma} \frac{1}{\sigma S}, k_t^H)$ is thus an $FER(H)$ on $[t, T]$, and because the P-density process of $\widehat{P}$ satisfies $R_{L \log L}(P)$ since $\frac{\mu}{\sigma}$ is bounded, this $FER(H)$ is even the unique $FER^\star(H)$ on $[t, T]$ by Theorem 2. The unique $FER^\star(0)$ (N^0, η^0, k_t^0) on $[t, T]$ is constructed analogously, with Ψ_t^H replaced by Ψ_t^0. This concludes the proof in view of Theorem 3. □

Remark 7 Proposition 5 can be extended to the more general framework of case (I) in Frei and Schweizer [10] where the correlation ϱ is no longer constant, but $\mathbb{Y}$-predictable with absolute value uniformly bounded away from one. The explicit form of the indifference value is then essentially preserved; see Theorem 2 of [10] for the precise formulation. This can also be proved with our methods here, but we only sketch the main steps for $t=0$ since the full details are a bit technical. First, one calls a triple (N^H,η^H,k_0^H) an *upper* (or *lower*) $FER^\star(H)$ if it has the properties of an $FER^\star(H)$, except that the equality sign in (7) is replaced by "$\geq$" (or "$\leq$"). One then shows that for an upper (lower) $FER^\star(H)$, (28) is satisfied with "$\leq$" ("$\geq$") instead of equality. In a third step, one defines constants

$$\overline{\delta}:=\sup_{s\in[0,T]}\left\|\frac{1}{1-|\varrho_s|^2}\right\|_{L^\infty(P)}\quad\text{and}\quad\underline{\delta}:=\inf_{s\in[0,T]}\frac{1}{\|1-|\varrho_s|^2\|_{L^\infty(P)}}$$

and finds, in the spirit of (64), $\mathbb{Y}$-predictable processes $\overline{\zeta}$ and $\underline{\zeta}$ such that

$$|\Psi_0^H|^{1/\overline{\delta}}=E_{\widehat{P}}[|\Psi_0^H|^{1/\overline{\delta}}]\mathscr{E}\Big(\int\overline{\zeta}\,\mathrm{d}\widehat{Y}\Big)_T\quad\text{and}\quad E_{\widehat{P}}\bigg[\int_0^T|\overline{\zeta}_s|^2\mathrm{d}s\bigg]<\infty,$$

with an analogous construction for $\underline{\zeta}$. For this one uses that $\widehat{Y}$ is $\mathbb{Y}$-adapted because ϱ is $\mathbb{Y}$-predictable. Similarly to the proof of Proposition 5, one shows that $(\overline{N}^H,\overline{\eta}^H,\overline{k}_0^H)$ is an upper $FER^\star(H)$, where $\overline{N}^H=\int\overline{\delta}\,\overline{\zeta}\sqrt{1-|\varrho|^2}\mathrm{d}\widehat{W}^\perp-\int\frac{\mu}{\sigma}\mathrm{d}W$, $\overline{\eta}^H=\frac{\overline{\delta}\varrho\overline{\zeta}}{\gamma}\frac{1}{\sigma S}+\frac{\mu}{\gamma\sigma}\frac{1}{\sigma S}$ and $\overline{k}_0^H=\frac{\overline{\delta}}{\gamma}\log E_{\widehat{P}}[|\Psi_0^H|^{1/\overline{\delta}}]$. A completely analogous result holds for $\underline{\delta}$. Therefore, one obtains

$$-\exp(-\gamma x_0+\gamma\underline{k}_0^H)\leq V_0^H(x_0)\leq-\exp(-\gamma x_0+\gamma\overline{k}_0^H)$$

by the above versions of (28). Because $\delta\mapsto\delta\log E_{\widehat{P}}[|\Psi_0^H|^{1/\delta}]$ is continuous on $[\underline{\delta},\overline{\delta}]$, interpolation then yields the existence of $\delta_0^H\in[\underline{\delta},\overline{\delta}]$ such that

$$V_0^H(x_0)=-\exp(-\gamma x_0)E_{\widehat{P}}[|\Psi_0^H|^{1/\delta_0^H}]^{\delta_0^H}.$$

Solving the implicit equation (3) with respect to h_0 finally gives an explicit expression for h_0.

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