7
Risk Management

Basically, risk management deals with the problem of protecting a portfolio or trading book against unexpected changes of market prices or other parameters. It therefore expresses the desire of a portfolio manager or trader to guarantee a minimum holding period return or to create a portfolio which helps him to fullfil specific liabilities over time. Risk management may help to avoid extreme events, to reduce the tracking error or even the trading costs. However, there are different possibilities for setting up a risk management or hedging process. The method which should be applied may well depend on the time horizon of the risk manager. If he is interested in controlling short-term risk, or if he would like to hedge against small movements in market prices, he may decide for a sensitivity-based risk management. This method is described in Section 7.1. If he has a longer time horizon and wants to be safe against large market movements he may decide for a downside risk management, which is discussed in Section 7.2.

7.1 Sensitivity-Based Risk Management

Sensitivity-based risk management deals with the problem of controlling a portfolio's sensitivity with respect to a given set of risk factors. It concentrates on hedging against small movements of the risk factors in a small period of time. This is of special interest to traders in charge of controlling the intraday or overnight market risk of their trading book. Section 7.1.1

gives a general definition of first- and second-order hedging which is applied to special risk measures such as the duration measure in Section 7.1.2 or the key-rate deltas and gammas in Section 7.1.3. For ease of exposition we omit the index for the specific daycount convention in writing the length of a time interval. The reader interested in more details on this topic refer to Section 5.1.

7.1.1 First- and Second-Order Hedging

Having defined the sensitivity measures of first- and second-order, we now discuss how these measures can be used for risk management or hedging purposes. To do so, let $V(\mathsf{F},\varphi)$ be the price of a portfolio $\varphi = (\varphi_1, ..., \varphi_n)$ of financial instruments or derivatives with prices $D_1(\mathsf{F}), ..., D_n(\mathsf{F})$ depending on the vector of risk factors $\mathsf{F} = (\mathsf{F}_1, ..., \mathsf{F}_m)$, i.e.

$$V(\varphi) = V(\mathsf{F},\varphi) = \sum_{i=1}^{n} \varphi_i \cdot D_i(\mathsf{F}),$$

and let $H_1(\mathsf{F}), ..., H_K(\mathsf{F})$, $K \in \mathbb{N}$, be the prices of the financial instruments which the trader or risk manager would allow for hedging purposes and which we will call hedge instruments. Furthermore, let $h = (h_1, ..., h_K)$ be a portfolio consisting of these hedge instruments, called a hedge portfolio, with a portfolio price given by

$$V(h) = V(\mathsf{F},h) = \sum_{k=1}^{K} h_k \cdot H_k(\mathsf{F}).$$

The first-order sensitivities of the portfolios φ and h are given by

$$\Delta_{\mathsf{F}_j}^{V(\varphi)} = \sum_{i=1}^{n} \varphi_i \cdot \Delta_{\mathsf{F}_j}^{D_i}(\mathsf{F})$$

and

$$\Delta_{\mathsf{F}_j}^{V(h)} = \sum_{k=1}^{K} h_k \cdot \Delta_{\mathsf{F}_j}^{H_k}(\mathsf{F}),\ j = 1, ..., m.$$

For a fixed vector $\alpha = (\alpha_1, ..., \alpha_m) \in [0,1]^m$, the idea behind the so-called *first-order hedging* is to find a portfolio $h^* = (h_1^*, ..., h_K^*)$, sometimes called the vector of the *first-order hedge ratios*, which solves the optimization problem

$$(P_1) \begin{cases} \sum_{j=1}^{m} \alpha_j \cdot \left(\Delta_{\mathsf{F}_j}^{V(\varphi)}(\mathsf{F}) - \Delta_{\mathsf{F}_j}^{V(h)}(\mathsf{F}) \right)^2 \to \min \\ h \in Z_1 \subseteq \mathbb{R}^K, \end{cases}$$

where Z_1 denotes the set of all possible hedge portfolios, which we assume to be set up by linear restrictions. This set may be equal to $\mathbb{R}^K$ if no trading restrictions are set by the trader, but there may be restrictions on the trading volume such as $s_{low} \leq h \leq s_{high}$ or even special corridors into which the risk manager or trader would like to drive the first-order sensitivities, i.e.

$$\Delta_{low} \leq \Delta_{\mathsf{F}}^{V(\varphi)}(\mathsf{F}) - \Delta_{\mathsf{F}}^{V(h)}(\mathsf{F}) \leq \Delta_{high}.$$

Such restrictions may be interesting especially for those first-order sensitivities for which α_j was set equal to 0 and which are therefore not included in the minimization process. Note that (P_1) is of the general form

$$(P)\begin{cases} h'Qh + c'h + d \to \min \\ \quad h \in Z \end{cases}$$

for suitable vectors c and d, a symmetric matrix $Q \in \mathbb{R}^{K \times K}$, and a set of possible portfolios Z set up by linear restrictions. Especially, Q in the case of problem (P_1) is given by

$$Q = (q_{kl})_{k,l=1,\dots,K} \quad \text{with} \quad q_{kl} := \sum_{j=1}^{m} \alpha_j \cdot \Delta_{\mathsf{F}_j}^{H_k}(\mathsf{F}) \cdot \Delta_{\mathsf{F}_j}^{H_l}(\mathsf{F}),\ k,l = 1,\dots,K.$$

The goal function is convex if and only if Q is positive semi-definite, it is strictly convex if and only if Q is positive definite. In either case, the corresponding optimization problem is called quadratic. It is well-known from the theory of non-linear optimization that the quadratic optimization problem (P) has a solution if Q is positive semi-definite, $Z \neq \emptyset$, and the goal function is bounded below. (P) has a unique solution if Q is positive definite and $Z \neq \emptyset$. Very often traders and risk managers are only interested in first-order hedging with respect to a single risk factor, i.e. $\alpha = e_j$ for some $j \in \{1, \dots, m\}$ and $Z_1 = \mathbb{R}^K$. We will refer to this special case as *single-factor first-order hedging.* In this case, and if enough instruments with an exposure in the corresponding risk factor are made available for hedging, the value of the goal function in the previous optimization problem is zero, i.e.

$$\Delta_{\mathsf{F}_j}^{V(\varphi)}(\mathsf{F}) = \Delta_{\mathsf{F}_j}^{V(h^*)}(\mathsf{F}) = \sum_{k=1}^{K} h_k^* \cdot \Delta_{\mathsf{F}_j}^{H_k}(\mathsf{F}). \tag{7.1}$$

The resulting hedged portfolio $(\varphi, -h^*)$ derived by adding a short position in the hedge portfolio h^* to the portfolio φ is called *first-order neutral* (with respect to risk factor F_j). The second-order sensitivities of the portfolios φ and h are given by

$$\Gamma_{\mathsf{F}_j\mathsf{F}_l}^{V(\varphi)}(\mathsf{F}) = \sum_{i=1}^{n} \varphi_i \cdot \Gamma_{\mathsf{F}_j\mathsf{F}_l}^{D_i}(\mathsf{F})$$

and

$$\Gamma^{V(h)}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F}) = \sum_{k=1}^{K} h_k \cdot \Gamma^{H_k}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F}), \; j,l = 1,...,m.$$

For a fixed matrix $\beta = \left(\beta_{jl}\right)_{j,l=1,...,m} \in [0,1]^{m\times m}$, the idea behind the so-called *second-order hedging* is to find a portfolio $h^* = (h_1^*, ..., h_K^*)$, sometimes called the vector of the *second-order hedge ratios*, which solves the optimization problem

$$(P_2)\begin{cases} \displaystyle\sum_{j=1}^{m}\sum_{l=1}^{m} \beta_{jl} \cdot \left(\Gamma^{V(\varphi)}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F}) - \Gamma^{V(h)}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F})\right)^2 \to \min \\ \quad h \in Z_2 \subseteq I\!R^K. \end{cases}$$

Again, we suppose that the set of all possible hedge portfolios Z_2 is set up by linear restrictions. One possible restriction may be equation (7.1) to ensure that the residual portfolio will have a first-order sensitivity of zero with respect to factor F_j and second-order sensitivities as low as possible. Traders and risk managers are very often interested in second-order hedging only with respect to a single risk factor, i.e. $\beta_{jj} = 1$ for some $j \in \{1, ..., m\}$ and $\beta_{jl} = 0$ for all other possible pairs (j,l) as well as $Z_2 = I\!R^K$. We will refer to this special case as *single-factor second-order hedging.* In this case, and if enough instruments with an exposure in the corresponding risk factor are made available for hedging, the value of the goal function in the previous optimization problem is zero, i.e.

$$\Gamma^{V(\varphi)}_{\mathsf{F}_j\mathsf{F}_j}(\mathsf{F}) = \Gamma^{V(h^*)}_{\mathsf{F}_j\mathsf{F}_j}(\mathsf{F}) = \sum_{k=1}^{K} h_k^* \cdot \Gamma^{H_k}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F}). \tag{7.2}$$

The resulting hedged portfolio $(\varphi, -h^*)$ is called *second-order neutral* (with respect to risk factor F_j).

Advanced risk management often includes first- and second-order hedging. In this case traders are not only interested in reducing the sensitivity of their portfolio to given risk factors but also would like to reduce the frequency of restructuring the portfolio after small market changes. There are a few possibilities for how this can be done. The first one is to minimize a combination of the first- and second-order sensitivities. For a fixed number $\lambda \in [0,1]$, the idea is to find a portfolio $h^* = (h_1^*, ..., h_K^*)$ which solves the optimization problem

$$(P_\lambda)\begin{cases} \lambda \cdot \displaystyle\sum_{j=1}^{m} \alpha_j \cdot \left(\Delta^{V(\varphi)}_{\mathsf{F}_i}(\mathsf{F}) - \Delta^{V(h)}_{\mathsf{F}_i}(\mathsf{F})\right)^2 \\ + (1-\lambda) \cdot \displaystyle\sum_{j=1}^{m}\sum_{l=1}^{m} \beta_{jl} \cdot \left(\Gamma^{V(\varphi)}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F}) - \Gamma^{V(h)}_{\mathsf{F}_j\mathsf{F}_l}(\mathsf{F})\right)^2 \quad \to \quad \min \\ \quad h \in Z_{2-\lambda}, \end{cases}$$

where the set $Z_{2-\lambda}$ of possible hedge portfolios is supposed to be set up by linear restrictions. Choosing $\lambda = 1$ the risk manager or trader is interested in first-order hedging only. Choosing $\lambda = 0$ as another extreme, he is interested in pure second-order hedging.

Let us look at a first *example*. In Section 6.1.1 we learned that the first-order sensitivity of the (static) coupon-bond futures price $F(t,T)$ with respect to the cheapest-to-deliver bond (CTD) with price $Bond(t, T_B^*, C^*)$, under the assumptions that the CTD bond doesn't change by a small change of the coupon-bond price, that there are no coupon payments in the time-period $[t,T]$, and that all hedge ratios are due to the corresponding notional amounts of the future and the CTD, is given by (see equation (6.4))

$$\Delta_{CTD}^{F}(\mathsf{F}) = \frac{1 + R_L(t,T) \cdot (T-t)}{Conv(T_B^*, C^*)}.$$

Since $\Delta_{CTD}^{CTD}(\mathsf{F}) = 1$, the first-order hedge ratio for hedging the future with the CTD to receive first-order neutrality, according to equation (7.1), is given by

$$\Delta_{CTD}^{F}(\mathsf{F}) = h_{CTD}^* \cdot \Delta_{CTD}^{CTD}(\mathsf{F}) = h_{CTD}^*.$$

The corresponding first-order hedge ratio for hedging the CTD with the future to receive first-order neutrality is given by

$$h_F^* = \frac{Conv(T_B^*, C^*)}{1 + R_L(t,T) \cdot (T-t)},$$

i.e. we have to buy h_F^* futures to hedge the CTD. Under the additional assumption that the repo rate $R_L(t,T)$ doesn't change if the price of the CTD changes[1], this is also the hedge ratio with respect to a changing zero-rate curve since, in this case, the future price-changes with changes of the zero-rate curve only by the price-changes of the CTD. Note that the hedge ratios have to be multiplied by the corresponding ratio of the notional amounts if these are different for the future and the CTD (see Section 7.1.2 for an example).

7.1.2 Duration-Based Hedging

Let us now drop the assumption that the repo rate $R_L(t,T)$ doesn't change with price-changes of the CTD, and assume that all price-changes are due to a parallel shift of the zero-rate curve. This is the assumption we made in Section 6.1.3 claiming that the zero rates of all maturities move by

$$\Delta R(t,T) := \Delta \mathsf{F}(t) \quad \text{for all } T \in [t, T^*]. \tag{7.3}$$

[1] Note, that this assumption is consistent with assumption 2 of the Black model, i.e. interest rates are supposed to be deterministic for discounting purposes.

Consequently, we assume that the linearly quoted repo rate moves according to a parallel shift of the continously quoted zero-rate curve. The continuously compounded equivalent $R_C(t,T)$ to the repo rate is implicitly defined by the equation

$$e^{R_C(t,T)\cdot(T-t)} = 1 + R_L(t,T)\cdot(T-t),$$

or, the other way round,

$$R_L(t,T) = \frac{1}{T-t}\cdot\left[e^{R_C(t,T)\cdot(T-t)} - 1\right].$$

Hence, a small change of $R_C(t,T)$ will change the repo rate by

$$\begin{aligned}\Delta^{R_L}(\mathsf{F}) &:= \tfrac{\partial}{\partial R_C(t,T)} R_L(t,T) = e^{R_C(t,T)\cdot(T-t)} \\ &= 1 + R_L(t,T)\cdot(T-t).\end{aligned}$$

Furthermore, using equation (6.4), we know that a small change of the CTD will change the (static) futures price by

$$\Delta^F_{CTD}(\mathsf{F}) = \frac{1 + R_L(t,T)\cdot(T-t)}{Conv(T_B^*, C^*)},$$

with (T_B^*, C^*) characterizing the cheapest-to-deliver bond. Denoting the first-order sensitivities of the (static) future price $F = F(t,T)$ and the cheapest-to-deliver bond price $CTD = CDT(t, T_B^*, C^*)$ with respect to a parallel shift of the zero-rate curve by $\Delta^F(\mathsf{F})$ and $\Delta^{CTD}(\mathsf{F})$, we know by equation (6.15) that

$$\Delta^{CTD}(\mathsf{F}) = -duration(t, T_B^*, C^*)\cdot CTD.$$

To derive $\Delta^F(\mathsf{F})$ let us now look at the (static) futures price, again assuming that the CTD bond doesn't change by a small change of the coupon-bond price, that there are no coupon payments in the time-period $[t,T]$, and that all hedge ratios are due to the corresponding notional amounts of the future and the CTD. Following equation (5.4) it is given by

$$F(t,T) = CTD\cdot\frac{1 + R_L(t,T)\cdot(T-t)}{Conv(T_B^*, C^*)} - \frac{Accrued(t_0, T, C^*)}{Conv(T_B^*, C^*)}.$$

Hence,

$$\begin{aligned}
\Delta^F(\mathsf{F}) &= \Delta^{CTD}(\mathsf{F}) \cdot \frac{1 + R_L(t,T)\cdot(T-t)}{Conv(T_B^*, C^*)} \\
&\quad + CTD \cdot \frac{1 + R_L(t,T)\cdot(T-t)}{Conv(T_B^*, C^*)} \cdot (T-t) \\
&= CTD \cdot \frac{1 + R_L(t,T)\cdot(T-t)}{Conv(T_B^*, C^*)} \cdot \\
&\quad \cdot \left[-duration(t, T_B^*, C^*) + T - t\right] \\
&= \left(F(t,T) + \frac{Accrued(t_0, T, C^*)}{Conv(T_B^*, C^*)}\right) \cdot \\
&\quad \cdot \left[-duration(t, T_B^*, C^*) + T - t\right] \\
&= F_{dirty}(t,T) \cdot \left[-duration(t, T_B^*, C^*) + T - t\right]
\end{aligned}$$

with

$$F_{dirty}(t,T) := F(t,T) + \frac{Accrued(t_0, T, C^*)}{Conv(T_B^*, C^*)}$$

denoting the so-called *dirty price of the future*. If we may suppose that $T - t \approx 0$, we know that $F_{dirty}(t,T) \approx \frac{CTD(t, T_B^*, C^*)}{Conv(T_B^*, C^*)}$ which leads us to

$$\begin{aligned}
\Delta^F(\mathsf{F}) &\approx -duration(t, T_B^*, C^*) \cdot F_{dirty}(t,T) \\
&\approx -duration(t, T_B^*, C^*) \cdot \tfrac{CTD(t, T_B^*, C^*)}{Conv(T_B^*, C^*)}.
\end{aligned}$$

Having made this preparatory work, we can now turn our interest to the problem of hedging a portfolio of coupon bonds with a coupon-bond future to receive first-order neutrality with respect to a parallel shift of the zero-rate curve. The result is summarized in the following lemma.

Lemma 7.1 (Duration-Based Hedge Ratios) *Let $V(\mathsf{F}, \varphi, t)$ be the dirty price of a portfolio $\varphi = (\varphi_1, ..., \varphi_n)$ of coupon bonds depending on the risk factor F at time $t \in [t_0, T]$. Furthermore, let $duration(\varphi, t)$ denote the duration and $\Delta^{V(\varphi)}(\mathsf{F}) = \Delta^{V(\mathsf{F},\varphi,t)}(\mathsf{F})$ denote the first-order sensitivity of the coupon-bond portfolio with respect to F at time $t \in [t_0, T]$. Under the assumption that the CTD bond doesn't change by a small change of the coupon-bond price and that there are no coupon payments in the time-period $[t, T]$, the first-order hedge ratio $h_F^*(t)$ at time $t \in [t_0, T]$ for hedging the coupon-bond portfiolio with the future to receive first-order neutrality is given by*

$$\begin{aligned}
h_F^*(t) &= \frac{\Delta^{V(\varphi)}(\mathsf{F})}{\Delta^F(\mathsf{F})} \\
&= \frac{-duration(\varphi, t) \cdot V(\mathsf{F}, \varphi, t)}{F_{dirty}(t,T) \cdot \left[-duration(t, T_B^*, C^*) + T - t\right]}
\end{aligned} \tag{7.4}$$

for $t \in [t_0, T]$ *and with* $CTD = CTD\left(t, T_B^*, C^*\right)$*. The hedge ratio* $h_F^*(t)$ *is sometimes called duration-based hedge ratio at time* $t \in [t_0, T]$*.*

In practice, risk and portfolio managers often use reasonable approximations of equation (7.4). These are summarized in the following corollary.

Corollary 7.2 *Let the assumptions of Lemma 7.1 be satisfied. Furthermore, let* $V_{clean}(\mathsf{F}, t, \varphi)$ *denote the clean price of the portfolio* φ *at time* $t \in [t_0, T]$*.*

a) If $T - t \approx 0$*, then*

$$\begin{aligned} h_F^*(t) &\approx \frac{duration\,(t,\varphi) \cdot V\left(\mathsf{F},\varphi,t\right)}{duration\left(t,T_B^*,C^*\right) \cdot F_{dirty}\left(t,T\right)} \\ &\approx \frac{duration\,(\varphi,t) \cdot V\left(\mathsf{F},\varphi,t\right) \cdot Conv\left(T_B^*,C^*\right)}{duration\left(t,T_B^*,C^*\right) \cdot CTD}. \end{aligned} \tag{7.5}$$

b) If, in addition to the assumptions of part a),

$$\frac{V\left(\mathsf{F},\varphi,t\right)}{F_{dirty}\left(t,T\right)} \approx \frac{V_{clean}\left(\mathsf{F},\varphi,t\right)}{F\left(t,T\right)},$$

then

$$h_F^*(t) \approx \frac{duration\,(\varphi,t) \cdot V_{clean}\left(\mathsf{F},\varphi,t\right)}{duration\left(t,T_B^*,C^*\right) \cdot F\left(t,T\right)}. \tag{7.6}$$

Especially equation (7.6) is very popular among risk and portfolio managers since durations, clean coupon bond and (clean) futures prices are directly available in the market. Also, up to today, Macaulay and modified duration is available via commercial software and information systems rather than the zero-rate based duration of equation (7.6). So risk and portfolio managers tend to use one of these instead of the zero-rate based duration. The following example shows this practical application of equation (7.6) using the Macaulay duration. The problems arising with this application are analyzed in the case study of Section 7.1.3. For the practical application, note that h_F^* was calculated under the assumption that the futures price is evaluated relative to the same notional amount as the CTD. If the notional N_F of the future and the notional N_{CTD} of the CTD do not coincide, the duration-based hedge ratio of equations (7.4), (7.5) or (7.6) has to be adjusted to

$$h_F^{trading} = \frac{N_{CTD}}{N_F} \cdot h_F^*. \tag{7.7}$$

Case Study (Hedging Bond Portfolios with Futures)

Let us consider the following coupon-bond portfolio $\varphi = (\varphi_1, ..., \varphi_7)$ with a notional amount of 90 Mio. Euro, a portfolio clean price of $V_{clean}(\varphi, t) = 88,740,750$ (prices are already multiplied with $\frac{N_{CTD}}{100}$) at October 20, 2000 (t), and a Macaulay duration of $duration_{Mac}(\varphi, t) = 5.41$ years consisting of the following coupon bonds:

Notional amount (in Mio.)	Coupon (in %)	Maturity
10	4.050	05/17/02
10	4.125	08/27/04
15	7.500	11/11/04
5	6.250	04/26/06
20	3.750	01/04/09
10	4.500	07/04/09
20	5.250	07/04/10

Usually, the portfolio is divided for a better duration-based hedging result which we do by splitting into the portfolios $\varphi^1 = (\varphi_1, ..., \varphi_4)$ with coupon bonds having a time to maturity of up to 6 years and $\varphi^2 = (\varphi_5, ..., \varphi_7)$ with coupon bonds having a time to maturity of more than 6 years. The corresponding portfolio prices at time t were $V_{clean}(\varphi^1, t) = 41,092,750$ and $V_{clean}(\varphi^2, t) = 47,648,000$ (prices are already multiplied with $\frac{N_{CTD}}{100}$), the Macaulay durations were $duration_{Mac}(\varphi^1, t) = 3.19$ years and $duration_{Mac}(\varphi^2, t) = 7.33$ years. The idea is to hedge portfolio φ^1 with the Bobl future and portfolio φ^2 with the Bund future. The price for the futures at time t were $F_{Bobl}(t) := F(t, T_{Bobl}) = 103.38$ for the Bobl and $F_{Bund}(t) := F(t, T_{Bund}) = 105.58$ for the Bund. The CTD for the Bobl future at that time was a 6.5% government bond with maturity time 10/14/05 and a duration of 4.43 years, the CTD for the Bund future was a 5.375% government bond with maturity time 01/04/10 and a duration of 7.21 years. At October 20, 2000 both futures were for a notional of $N_F = 100,000$ Euro. Using equation (7.6) and the Macaulay duration as explained above, we get the following hedge ratios at time t for the Bobl and Bund futures:

$$\begin{aligned} h_{Bobl}^{trading}(t) &\approx \frac{duration_{Mac}(\varphi^1, t) \cdot V_{clean}(\varphi^1, t)}{\frac{N_{Bobl}}{100} \cdot duration\left(t, T^*_{CTD(Bobl)}, C^*_{CTD(Bobl)}\right) \cdot F_{Bobl}(t)} \\ &= \frac{3.19 \cdot 41,092,750}{\frac{100,000}{100} \cdot 4.43 \cdot 103.38} = 286.23 \approx 286 \end{aligned}$$

and

$$h_{Bund}^{trading}(t) \approx \frac{duration_{Mac}\left(\varphi^2,t\right)\cdot V_{clean}\left(\varphi^2,t\right)}{\frac{N_{Bund}}{100}\cdot duration\left(t,T^*_{CTD(Bund)},C^*_{CTD(Bund)}\right)\cdot F_{Bund}(t)}$$
$$= \frac{7.33\cdot 47,648,000}{1,000\cdot 7.21\cdot 105.58} = 458.81 \approx 459.$$

So we are hedging the coupon-bond portfolio by selling 286 Bobl futures and 459 Bund futures. □

Sometimes it is also interesting to duration-hedge a coupon-bond portfolio with other coupon bonds. For this reason, let $V(\mathsf{F},\varphi,t)$ and $duration(\varphi,t)$ be the price and duration of a portfolio $\varphi=(\varphi_1,...,\varphi_n)$ of coupon bonds at time $t\in[t_0,T]$. Furthermore, let $h=(h_1,...,h_K)$ be a portfolio consisting of the coupon bonds which are available for hedging and which we will briefly denote by hedging coupon bonds. Let $H_k(\mathsf{F},t)=Bond\left(t,T_B^k,C^k\right)$ and $duration\left(t,T_B^k,C^k\right)$, $k=1,...,K\in I\!N$, be the prices and durations of these hedge instruments and

$$V(h,t)=V(\mathsf{F},h,t)=\sum_{k=1}^{K}h_k\cdot H_k(\mathsf{F},t) \text{ and } duration(h,t)$$

be the price and duration of the hedge portfolio as given in equation (6.16). Then the condition for the first-order hedge ratios $h^*=h^*(t)$ with $h^*=(h_1^*,...,h_K^*)$ for hedging the coupon-bond portfolio with the hedging coupon bonds to receive first-order neutrality at time t is given by

$$\Delta^{V(\varphi)}(\mathsf{F})=\Delta^{V(h^*)}(\mathsf{F})=\sum_{k=1}^{K}h_k^*\cdot\Delta^{H_k}(\mathsf{F}), \tag{7.8}$$

or equivalently

$$-duration(\varphi,t)\cdot V(\mathsf{F},\varphi,t)=-duration(h^*,t)\cdot V(\mathsf{F},h^*,t).$$

If $K=1$ with $T_B:=T_B^1$ and $C:=C^1$, this is

$$-duration(\varphi,t)\cdot V(\mathsf{F},\varphi,t)=-duration(t,T_B,C)\cdot h^*\cdot Bond(t,T_B,C),$$

or

$$h^*=\frac{duration(\varphi,t)\cdot V(\mathsf{F},\varphi,t)}{duration(t,T_B,C)\cdot Bond(t,T_B,C)}.$$

If $K>1$, there is more than one possibility, and further equations have to be added, such as

$$duration(\varphi,t)=duration(h^*,t).$$

In this case we need a minimum of two coupon bonds for hedging. For a number of two hedging coupon bonds we always get a combination of a coupon bond with a shorter and a coupon bond with a longer duration than that of the portfolio φ. We will return to hedging coupon-bond portfolios with coupon bonds in the next section.

7.1.3 Key-Rate Delta and Gamma Hedging

In Section 6.1.4 we showed that the key-rate deltas are a natural generalization of the duration concept. To realize this concept, the time to maturity interval $[0, T^* - t]$ was divided into $m \in I\!N$ non-overlapping subintervals $KB_1, ..., KB_m$ called the key-rate buckets. At time t, all zero rates having a time to maturity within the same bucket KB_j are supposed to move by exactly the same amount $\Delta\mathsf{F}_j(t)$, $j = 1, ..., m$. If $D(R(t), t)$ denotes the price of a financial instrument or derivative depending on (some elements of) the vector of zero rates $R(t) := (R(t, T_1), ..., R(t, T_n))'$ and time $t \in [t_0, T^*]$ with $t_0 \leq t \leq T_1 < \cdots < T_n \leq T^*$, the key-rate delta $\Delta^D_{KB_j}(R)$ is defined to be the first-order sensitivity of the derivatives price with respect to a small parallel shift of the zero-rate curve within key-rate bucket KB_j, $j = 1, ..., m$, and all other zero rates unchanged. The corresponding second-order sensitivity, the key-rate gamma with respect to the key-rate buckets KB_j and KB_l, $j, l = 1, ..., m$, is denoted by $\Gamma^D_{KB_j, KB_l}(R)$. Using these definitions, the approximate price-change $\Delta D(R)$ of a derivative, depending on small changes of the risk factors F, is given by

$$\Delta D(R) \approx \sum_{j=1}^{m} \Delta^D_{KB_j}(R) \cdot \Delta\mathsf{F}_j + \sum_{j=1}^{m}\sum_{l=1}^{m} \Gamma^D_{KB_j KB_l}(R) \cdot \Delta\mathsf{F}_j \cdot \Delta\mathsf{F}_l.$$

The price-change $\Delta V(\varphi, R)$ of the portfolio $\varphi = (\varphi_1, ..., \varphi_n)$ of financial instruments or derivatives with prices $D_1(R), ..., D_n(R)$ depending on the vector of zero rates R which is supposed to include all zero rates on which the derivatives may depend is approximately given by

$$\Delta V(\varphi, R) \approx \sum_{j=1}^{m} \Delta^{V(\varphi)}_{KB_j}(R) \cdot \Delta\mathsf{F}_j + \sum_{j=1}^{m}\sum_{l=1}^{m} \Gamma^{V(\varphi)}_{KB_j KB_l}(R) \cdot \Delta\mathsf{F}_j \cdot \Delta\mathsf{F}_l,$$

where $V(\varphi, R)$ denotes the price of the portfolio, the key-rate deltas of the portfolio are given by

$$\Delta^{V(\varphi)}_{KB_j}(R) = \sum_{i=1}^{n} \varphi_i \cdot \Delta^{D_i}_{KB_j}(R), \; j = 1, ..., m,$$

and the key-rate gammas of the portfolio are given by

$$\Gamma^{V(\varphi)}_{KB_j KB_l}(R) = \sum_{i=1}^{n} \varphi_i \cdot \Gamma^{D_i}_{KB_j KB_l}(R), \; j, l = 1, ..., m.$$

Let $H_1(R), ..., H_K(R)$, $K \in \mathbb{N}$, be the prices of the hedge instruments with key-rate deltas and gammas denoted by $\Delta_{KB_j}^{H_k}(R)$ and $\Gamma_{KB_jKB_l}^{H_k}(R)$, $j, l = 1, ..., m$, $k = 1, ..., K$. Furthermore, let $h = (h_1, ..., h_K)$ be the hedge portfolio with price $V(h) = V(h, R)$ and key-rate delta and gamma given by $\Delta_{KB_j}^{V(h)}(R)$ and $\Gamma_{KB_jKB_l}^{V(h)}(R)$, $j, l = 1, ..., m$. For a fixed vector $\alpha = (\alpha_1, ..., \alpha_m) \in [0,1]^m$, the so-called *key-rate delta hedging* is the search for a portfolio $h^* = (h_1^*, ..., h_K^*)$, sometimes called the vector of the *key-rate delta hedge ratios*, which solves the optimization problem

$$(P_1) \begin{cases} \sum_{j=1}^{m} \alpha_j \cdot \left(\Delta_{KB_j}^{V(\varphi)}(R) - \Delta_{KB_j}^{V(h)}(R)\right)^2 \to \min \\ h \in Z_1 \subseteq \mathbb{R}^K, \end{cases}$$

where Z_1 denotes the set of all possible hedge portfolios which we assume to be set up by linear restrictions. For a fixed matrix $\beta = \left(\beta_{jl}\right)_{j,l=1,...,m} \in [0,1]^{m \times m}$, the corresponding *key-rate gamma hedging* is the search for a portfolio $h^* = (h_1^*, ..., h_K^*)$, sometimes called the vector of the *key-rate gamma hedge ratios*, which solves the quadratic optimization problem

$$(P_2) \begin{cases} \sum_{j=1}^{m} \sum_{l=1}^{m} \beta_{jl} \cdot \left(\Gamma_{KB_jKB_l}^{V(\varphi)}(R) - \Gamma_{KB_jKB_l}^{V(h)}(R)\right)^2 \to \min \\ h \in Z_2 \subseteq \mathbb{R}^K. \end{cases}$$

Again, we suppose that the set of all possible hedge portfolios Z_2 is set up by linear restrictions. Combinations are possible as already mentioned in Section 7.1.1. Nevertheless, because of the complexity of the resulting optimization problems, key-rate delta hedging plays the dominant role in practice.

Case Study (Key-Rate Delta Hedging)[2]

In the *Duration-Based Hedging with Futures* case study of Section 7.1.2 we hedged, at time $t = 10/20/00$, a coupon-bond portfolio with a notional amount of 90 Mio. Euro using Bobl and Bund futures. We did this, using the simplest approximation of Corollary 7.2b), by selling $h_1^0 = 286$ Bobl and $h_2^0 = 459$ Bund futures which is a notional of 74.5 Mio. Euro. According to the classification of the coupon bonds into two maturity segments in this case study we now define the key-rate buckets $KB_1^A := [0M, 6Y]$ and $KB_2^A := (6Y, 10Y]$. The zero-rate curve at time t and the key-rate deltas of the duration hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^0, -h_2^0\right)$ is shown[3] in

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[2] All calculations and optimizations were done using the software tool Risk *Advisor* from risklab *germany*.

[3] Remember, that we plot the key rate deltas with respect to an increase of the corresponding zero rates by $1bp$.

figures 7.1-7.2. The two deltas do not completely net out because we split the portfolio, and so do not correctly consider coupon payments of the longer maturity coupon bonds which fall in the first key-rate bucket. Furthermore, the duration-based hedging formula of Corollary 7.2b) is just an approximation.

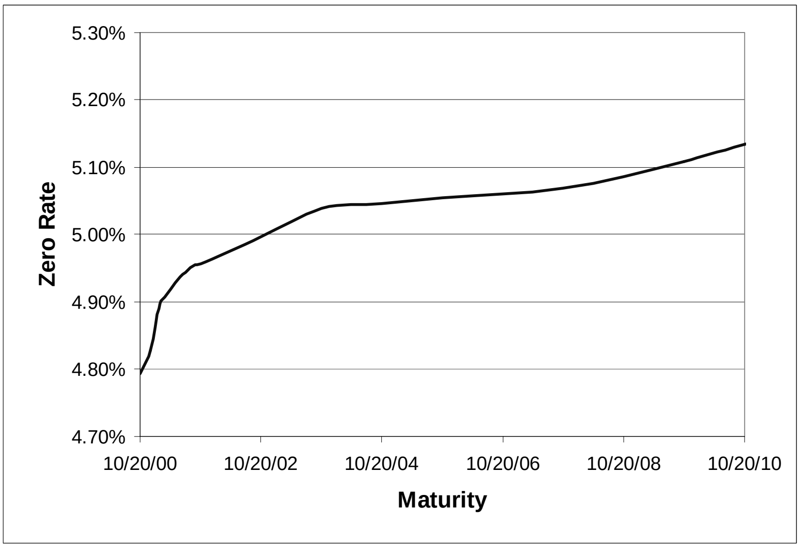

FIGURE 7.1. Continuous zero rate curve derived from German government bonds on October 20, 2000

On the other hand we apply the optimization problem (P_1) doing a key-rate delta hedging with respect to the same key-rate buckets and instruments. The optimal solution of this problem is given by selling $h_1^A = 360$ Bobl and $h_2^A = 442$ Bund futures which is a notional of 80.18 Mio. Euro. The corresponding key-rate deltas of the hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^A, -h_2^A\right)$ in figure 7.3 are both close to zero and by far smaller than those of portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^0, -h_2^0\right)$.

However, this is not the full picture. Let us consider the key-rate buckets $KB_1^B := [0M, 3Y]$, $KB_2^B := (3Y, 6Y]$, $KB_3^B := (6Y, 8Y]$, and $KB_4^B := (8Y, 10Y]$ and let us examine the corresponding key-rate deltas of portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^A, -h_2^A\right)$. The result is plotted in figure 7.4 and shows the risk inherent in what we thought a well hedged portfolio. It is indeed well hedged under the assumption that the zero-rate curve moves by parallel shifts only in the key-rate buckets KB_1^A and KB_2^A when the key-rate deltas of buckets KB_1^B and KB_2^B as well as those of buckets KB_3^B and KB_4^B net out. The hedge may be rather bad if the yield curve twists.

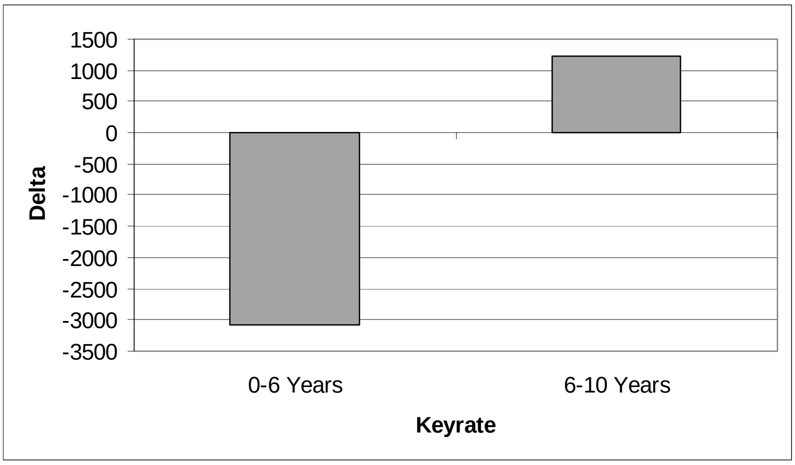

FIGURE 7.2. Key rate deltas of the duration hedged portfolio $(\varphi_1, ..., \varphi_7, -h_1^0, -h_2^0)$ with respect to the key rate buckets KB_1^A and KB_2^A

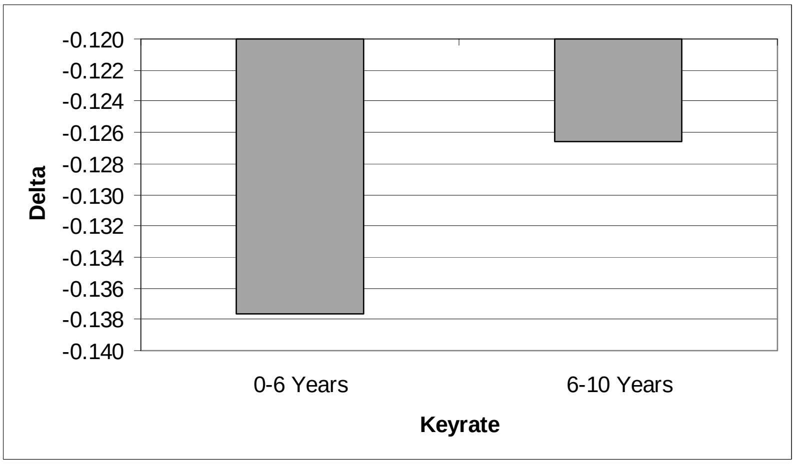

FIGURE 7.3. Key rate deltas of the key rate hedged portfolio $(\varphi_1, ..., \varphi_7, -h_1^A, -h_2^A)$ with respect to the key rate buckets KB_1^A and KB_2^A

One step further, let us use a 8% coupon bond with maturity time 01/21/02 and a 6% coupon bond with maturity time 07/04/07 as hedge instruments H_3 and H_4 and apply optimization problem (P_1) with respect to the key-rate buckets KB_1^B, KB_2^B, KB_3^B, and KB_4^B. The optimal solution is a hedge portfolio of $h_1^B = 259$ Bobl futures, $h_2^B = 445$ Bund futures which is a notional of 70.4 Mio., a notional of $h_3^B = -0.42$ Mio. of H_3, and a notional of $h_4^B = 33.62$ Mio. of H_4. This is a total notional amount of 103.6 Mio., i.e. the notional amount of our hedge portfolios increases the finer we hedge. Nevertheless, the risk numbers of the key-rate hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^B, -h_2^B, -h_3^B, -h_4^B\right)$, shown in figure 7.5, look much better than those of the hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^A, -h_2^A\right)$ as shown in figure 7.4.

Let us finally compare the risk of the starting coupon-bond portfolio with that of the key-rate hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^B, -h_2^B, -h_3^B, -h_4^B\right)$ for the yearly key-rate buckets $KB_1^C := [0M, 1Y], KB_2^C := (1Y, 2Y], ...,$ $KB_{10}^C := (9Y, 10Y]$ as shown in figures 7.6-7.7. Notice the typically negative key-rate deltas of the long coupon-bond portfolio in figure 7.6, due to the fact that coupon-bond prices fall if we increase the zero rates by $1bp$. We see that the main risk is concentrated in the key-rate buckets $KB_4^C, ..., KB_6^C$, corresponding to time to maturities from 3 to 6 years, and in the key-rate buckets KB_9^C and KB_{10}^C, corresponding to time to maturities from 8 to 10 years. As the analysis of the RiskMetrics^{TM} correlations in Section 6.4 showed, it is rather plausible to assume that the zero-rate curve in each of these two time to maturity segments will move by parallel shifts. So we can assume that the key-rate deltas will net out making the hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^B, -h_2^B, -h_3^B, -h_4^B\right)$ a rather good one. We could now add additional conditions such as a limit for the notional amount of the hedged portfolio to meet more specific needs of the risk controller or portfolio manager. □

One of the most important applications in portfolio management is the derivation of a portfolio that mirrors a given index portfolio. Such a portfolio is also called a *tracking portfolio* and the process of managing a portfolio to duplicate the index over time is called *index tracking*. Usually this is done by adjusting the duration or the key-rate delta of the tracking portfolio with respect to a single key-rate bucket to match that of the index. No wonder that the results can be very disappointing if the zero-rate movements are non-parallel. Therefore, we dedicated the following case study to compare the tracking portfolios for the J.P. Morgan government bond index Germany (JPMGBG), briefly denoted by J.P. Morgan index derived by a key-rate delta hedge with one (consistent with duration-based hedging) and with ten key-rate buckets.

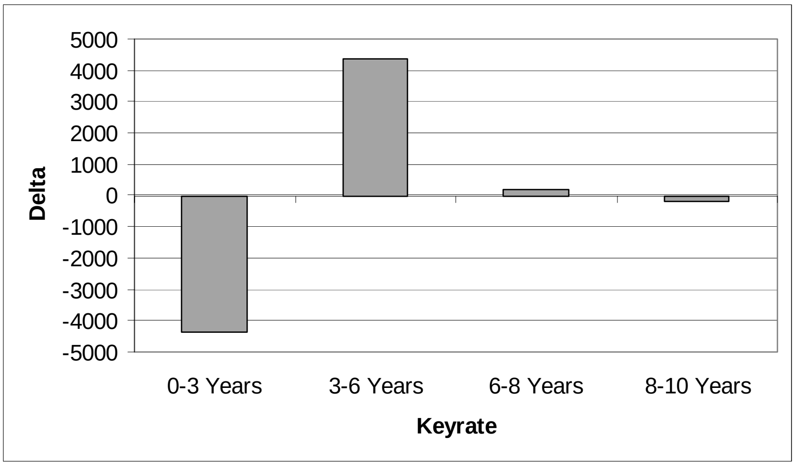

FIGURE 7.4. Key rate deltas of the key rate hedged portfolio $(\varphi_1, ..., \varphi_7, -h_1^A, -h_2^A)$ with respect to the key rate buckets KB_1^B, KB_2^B, KB_3^B, and KB_4^B

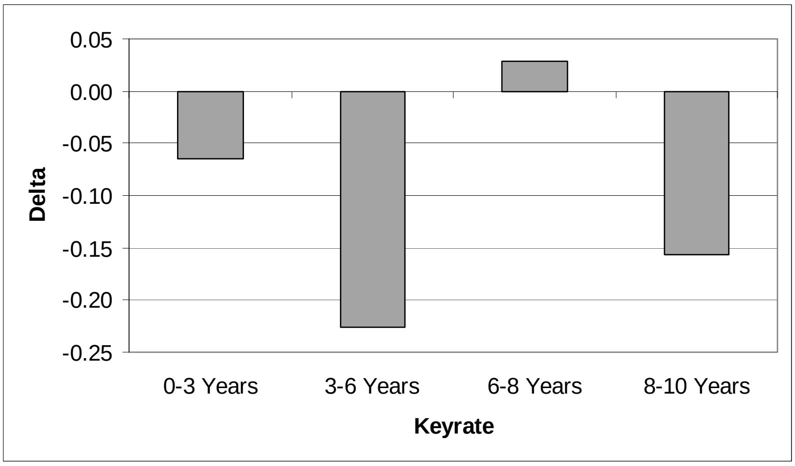

FIGURE 7.5. Key rate deltas of the key rate hedged portfolio $(\varphi_1, ..., \varphi_7, -h_1^B, -h_2^B, -h_3^B, -h_4^B)$ with respect to the key rate buckets KB_1^B, KB_2^B, KB_3^B, and KB_4^B

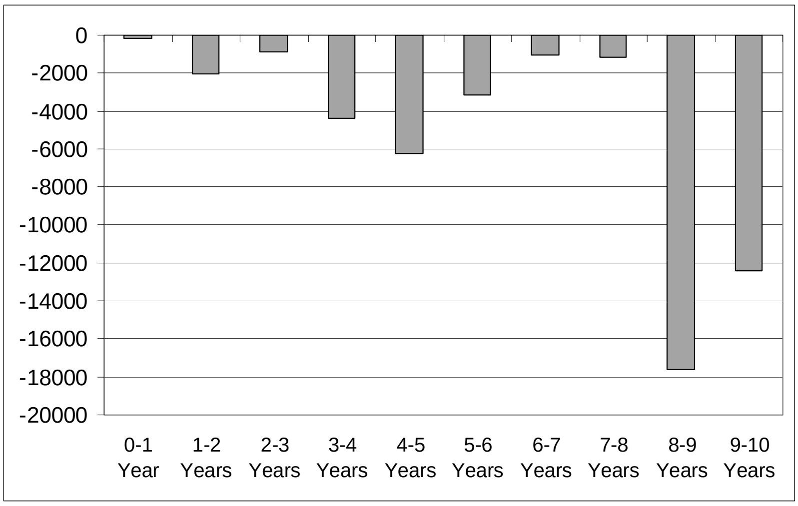

FIGURE 7.6. Key rate deltas of the initial bond portfolio $(\varphi_1, ..., \varphi_7)$ with respect to the key rate buckets $KB_1^C, KB_2^C, ..., KB_{10}^C$

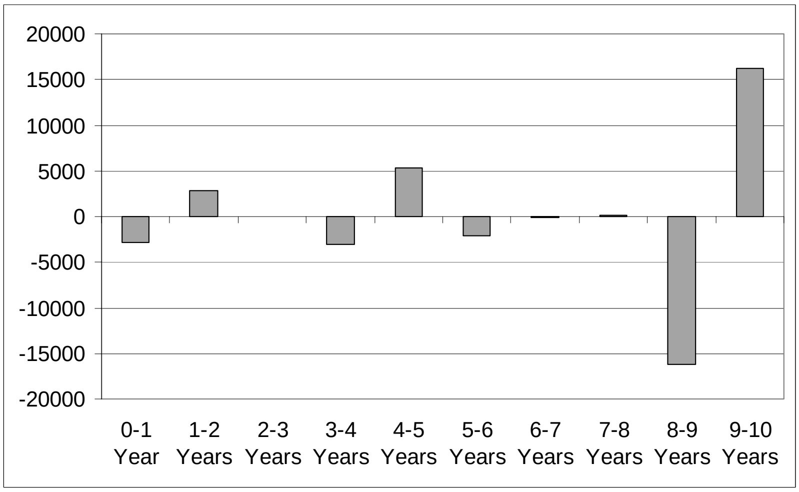

FIGURE 7.7. Key rate deltas of the key rate hedged portfolio $\left(\varphi_1, ..., \varphi_7, -h_1^B, -h_2^B, -h_3^B, -h_4^B\right)$ with respect to the key rate buckets $KB_1^C, KB_2^C, ..., KB_{10}^C$

Case Study (Index Tracking)[4]

Starting in April 1998, the J.P. Morgan index was hedged once a month. First this was done with respect to one key-rate bucket from 0 to 10 years due to a parallel shift of the yield curve, second with respect to the ten key-rate buckets $KB_1^C, KB_2^C, ..., KB_{10}^C$. In addition we claimed that the fair or clean prices of the J.P. Morgan index and those of the hedge portfolios should be equal. The corresponding hedge concepts are referred to as (fair price) duration tracking for the case of one key-rate bucket and (fair price) key-rate tracking for the case of ten key-rate buckets. The portfolios are compared after one month and then readjusted. Figure 7.8 shows the price behaviour of the duration tracking portfolio compared to the J.P. Morgan index. This happened while the correponding zero-rate curves changed as plotted in figure 7.9.

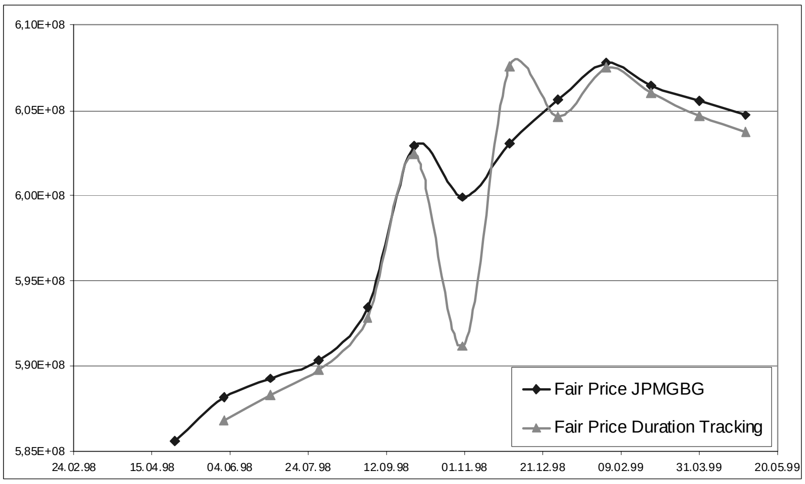

FIGURE 7.8. Price behaviour of the duration tracking portfolio compared to the J.P. Morgan index.

At the very beginning the duration tracking did quite well until the "big surprise" in October 1998 opened the eyes by a significant underperformance compared to the J.P. Morgan index which is shown numerically in the following table.

Time	J.P. Morgan	Duration tracking	change
8/31-09/30/98	$9,486,111$	$9,021,417$	parallel
9/30-10/31/98	$-3,031,613$	$-11,749,764$	twist

[4]All calculations and optimizations were done using the software tool Risk *Advisor* from risklab *germany*.

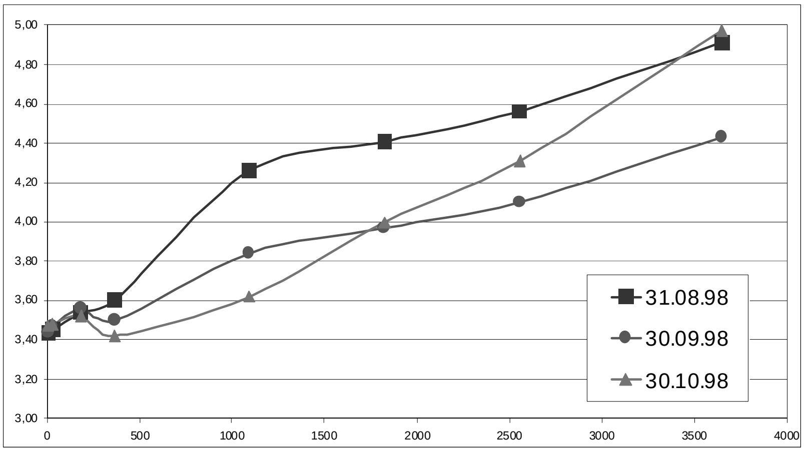

FIGURE 7.9. Changes of the zero rate curves.

Where did this underperformance in October 1998 come from? To answer this question let us dip into September 30, 1998. If we compare the price value of a basis point (see Section 6.1.3) of $-255,325.69$ for the J.P. Morgan index and $-255,326.21$ for the duration tracking portfolio at a clean price of $585,583,106.07$ for the first and $585,583,179.57$ for the latter we would have not expected such a development. This is underlined by the key-rate delta picture in figures 7.10-7.11 plotted by their days to maturity.

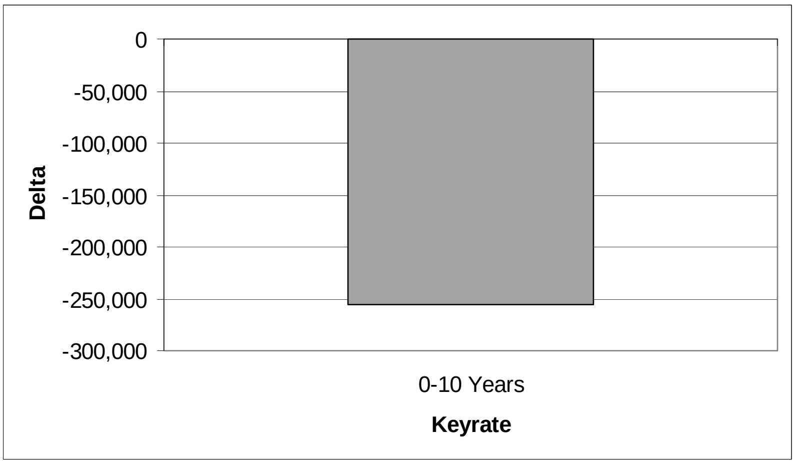

FIGURE 7.10. Key rate delta of the J.P. Morgan index with respect to one key rate bucket.

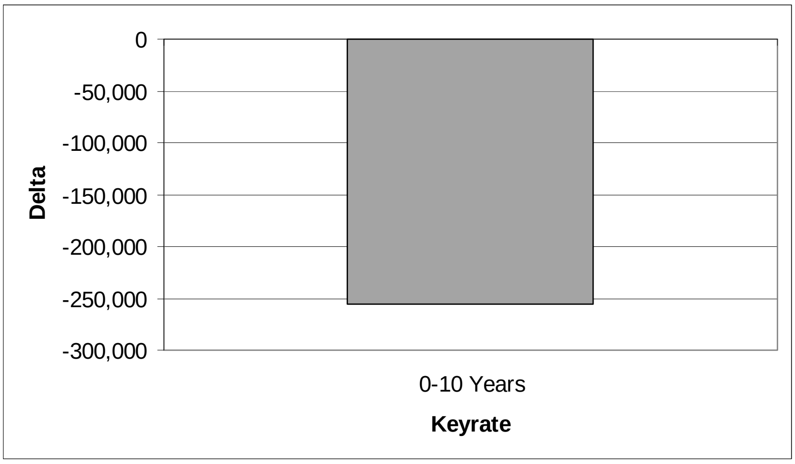

FIGURE 7.11. Key rate delta of the duration tracking portfolio with respect to one key rate bucket

However, with a little more insight, given by the key-rate delta pictures of figures 7.12-7.13 for a number of 10 key-rate buckets, we see that both portfolios have a completely different risk design.

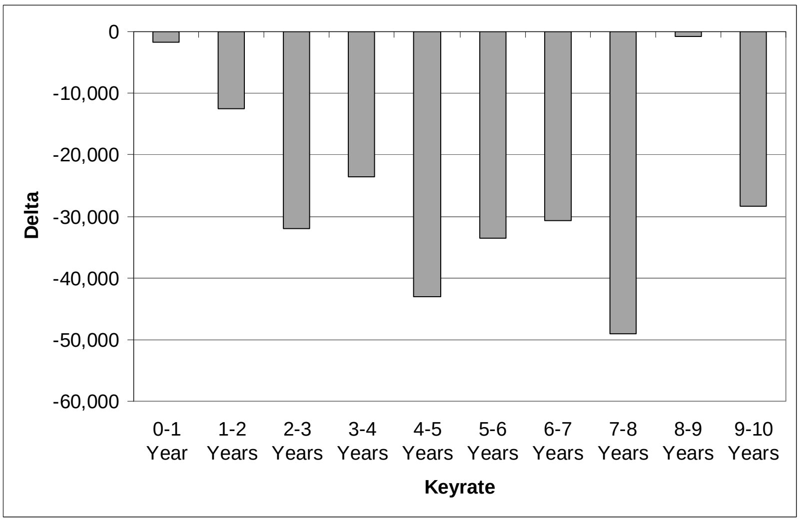

FIGURE 7.12. Key rate deltas of the J.P. Morgan index with respect to ten key rate buckets

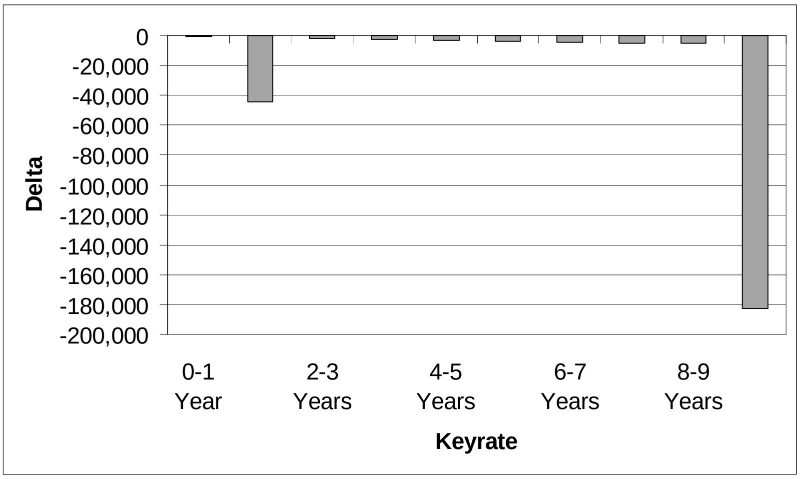

FIGURE 7.13. Key rate deltas of the duration tracking portfolio with respect to ten key rate buckets

More dramatically this is shown in figure 7.14, looking at the key-rate deltas of the portfolio derived by going long the duration tracking portfolio and short the J.P. Morgan index. We can see that the duration tracking portfolio extremely overweights the last maturity segment, resulting in a negative key-rate delta. A zero-rate curve moving up in this last maturity bucket may therefore have led to the underperformance we observed. And indeed, as already shown in figure 7.9, the zero-rate curve did increase in this bucket combined with a twist of the whole zero-rate curve.

If we use all ten key-rate buckets to solve hedging problem (P_1), the key-rate deltas of the resulting key-rate tracking portfolio pretty much equal those of the J.P. Morgan index (see figure 7.15). If we examine the portfolio resulting from going long the key-rate tracking portfolio and short the J.P. Morgan index we see that the key-rate deltas are much smaller than those of figure 7.14, especially in the last segment (see figure 7.16). Consequently, the corresponding price behaviour of the key-rate tracking portfolio is much closer to that of the J.P. Morgan index than that of the duration tracking portfolio which is shown in figure 7.17. As we can see, the key-rate tracking portfolio rather smoothly tracks the index, almost perfect compared to the duration tracking portfolio. This is documented by the tracking error, which is defined to be the square root of the sum of the squared deviations of the index price and the price of the corresponding tracking portfolio. It is calculated as $2,926,855$ or 0.50% for the duration tracking portfolio and $331,102$ or 0.05% for the key-rate tracking portfolio which is only 10% of the first value. □

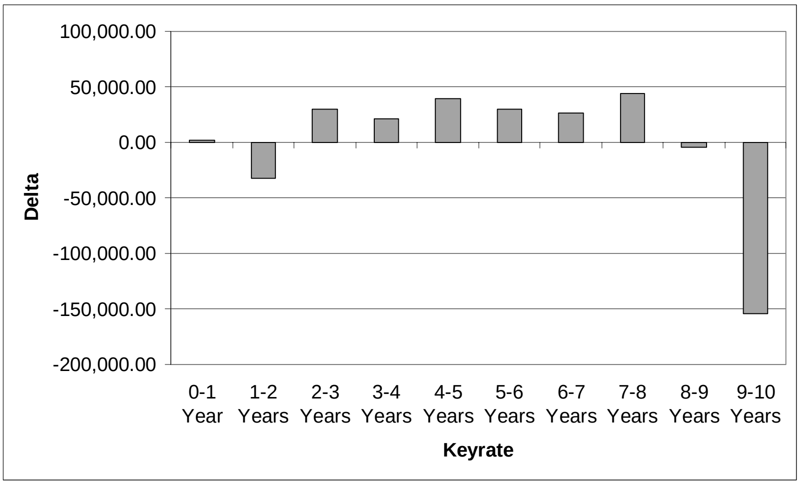

FIGURE 7.14. Key rate deltas of the portfolio resulting from going long the duration tracking portfolio and short the J.P. Morgan index with respect to ten key rate buckets

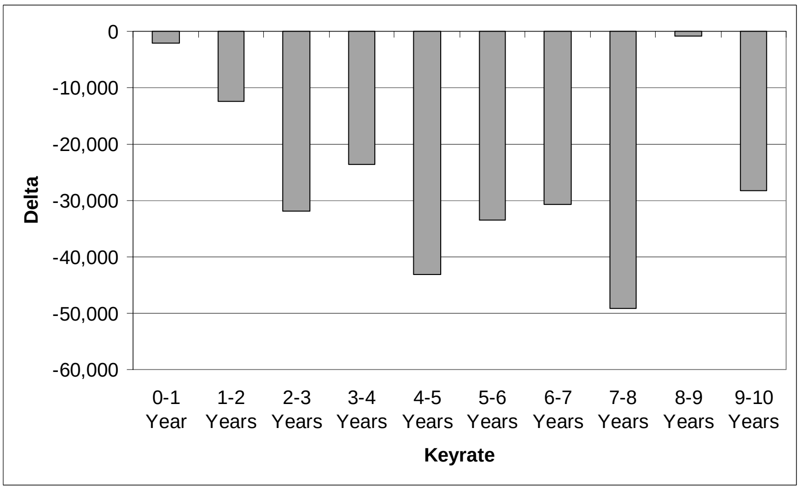

FIGURE 7.15. Key rate deltas of the key rate tracking portfolio with respect to ten key rate buckets

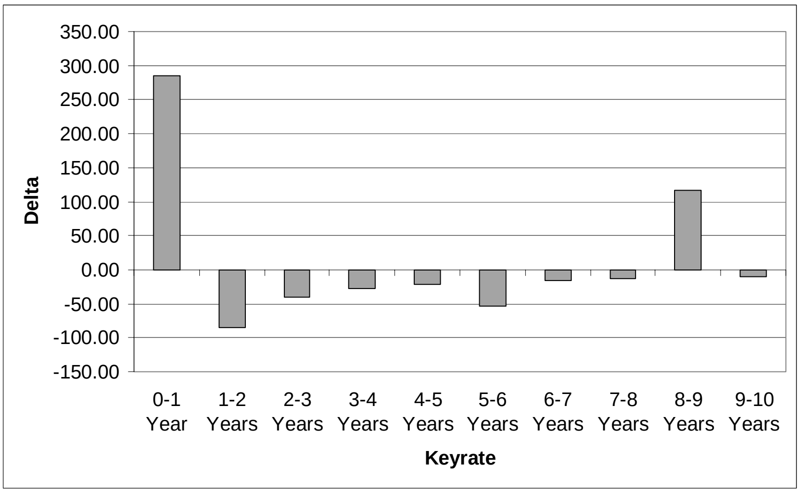

FIGURE 7.16. Key rate deltas of the portfolio resulting from going long the key rate tracking portfolio and short the J.P. Morgan index with respect to ten key rate buckets

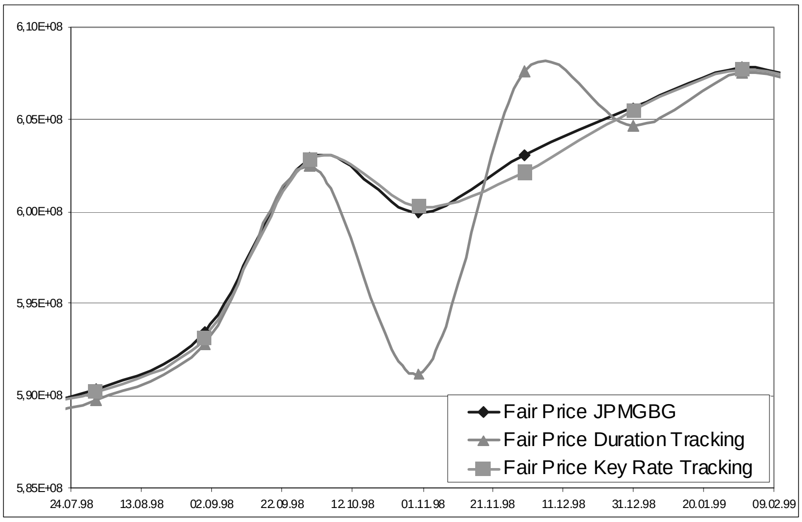

FIGURE 7.17. Price behaviour of the key rate tracking portfolio compared to the duration tracking portfolio and the J.P. Morgan index

Note that we optimized the tracking portfolios of the previous example without any linear restrictions but the price equality. Nevertheless, it could be interesting to add other restrictions such as a limit on the hedging volume or limits on the investment in instruments with a maturity falling in specific key-rate buckets to control the time structure of the portfolio. It could also be interesting to limit the sensitivity with respect to other risk factors such as the theta or vega, which could be easily implemented as shown in Section 7.1.1.

Another area of increasing interest is protecting a given portfolio φ of loans, considered to play the role of the index portfolio, with a portfolio h of coupon bonds, considered as playing the role of the tracking portfolio, against the risk of changing interest rates. This is already part of the so-called *asset liability management*. If we claim that both assets and liabilities have identical key-rate deltas, the price-change of the residual portfolio is approximately given by

$$\sum_{j=1}^{m}\sum_{l=1}^{m}\left(\Gamma_{KB_jKB_l}^{V(\varphi)}(R)-\Gamma_{KB_jKB_l}^{V(h)}(R)\right)\cdot\Delta\mathsf{F}_j\cdot\Delta\mathsf{F}_l.$$

Let us suppose that we model loans such as coupon bonds, and remember that coupon bonds have gamma exposure only in the diagonal elements of the gamma matrix. Then this price-change is equal to

$$\sum_{j=1}^{m}\left(\Gamma_{KB_jKB_j}^{V(\varphi)}(R)-\Gamma_{KB_jKB_j}^{V(h)}(R)\right)\cdot\left(\Delta\mathsf{F}_j\right)^2.$$

If we claim that the key-rate gammas $\Gamma_{KB_jKB_j}^{V(h)}(R)$ of the assets are always larger than the key-rate gammas $\Gamma_{KB_jKB_j}^{V(\varphi)}(R)$ of the liability side, i.e.

$$\Gamma_{KB_jKB_j}^{V(h)}(R)\geq\Gamma_{KB_jKB_j}^{V(\varphi)}(R),\text{ for all } j=1,...,m,$$

the approximate price-change of the residual portfolio will always be negative. In other words, an increase in price because of changing zero rates in any of the key-rate buckets will always be greater, a decrease in price will always be less on the asset side than that on the liability side. This generalizes the restriction that, under a parallel movement of the zero-rate curve, the convexity of the asset side should always be larger than the convexity of the liability side which can be found, e.g., in Dahl [Dah93].

Having discussed different possibilities for setting restrictions, there is also the possibility of changing the goal function. Under limited first- or second-order sensitivities it could be interesting to minimize the transaction costs or to maximize the expected return of a portfolio. The latter is of special interest for portfolio managers, especially when the planning horizon is long-term rather than short-term. Since this fits the background of downside risk measures, we will discuss downside risk management with the intention of maximizing the expected return of a portfolio in the next section.

7.2 Downside Risk Management

In this section we deal with the problem of managing the risk of large movements of the risk factors or interest rates in a possibly longer period of time. This problem especially appears in portfolio management when the portfolio manager wants to avoid the return of his portfolio falling below some given benchmark return. This problem is adressed in Section 7.2.1. On the other hand, the portfolio manager or trader may be controlled by a limit set on the value at risk of his portfolio therefore trying to be safe against extreme events. We will present a solution to this problem in Section 7.2.2.

7.2.1 Risk Management Based on Lower Partial Moments

The process of performing an optimal asset allocation basically deals with the problem of finding a portfolio that maximizes the expected utility of the investor or portfolio manager. In other words, the portfolio manager aims to choose a portfolio with a distribution function that maximizes the *expected utility*. As long as it is supposed that the returns of the portfolio assets follow a normal distribution, the return distribution of any portfolio considered will also be normal. In this case, as is done throughout the traditional portfolio theory introduced by Markowitz [Mar52] and Sharpe [Sha64], the problem of finding an expected utility-maximizing portfolio or distribution function for a risk-averse trader or portfolio manager, represented by a concave utility function, can be restricted to finding an optimal combination of the two parameters mean and variance. This dramatically simplifies the whole asset allocation process and is known as *mean-variance analysis*. It is the aim of the portfolio manager to find a portfolio that maximizes his expected return under a given risk level or a portfolio that minimizes his risk under a given return level. Risk in this case is measured by the variance of the portfolio return.

Unfortunately, selection rules based on the two parameters mean and variance are of limited generality. Roughly speaking they are optimal if the utility function is quadratic or if it is concave and the return distribution is