Bounded Perturbations of the Resolvent Operators Associated to the \( \mathcal{F} \)-Spectrum

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Abstract. Recently, we have introduced the \( \mathcal{F} \)-functional calculus and the SC-functional calculus. Our theory can be developed for operators of the form \( T = T_0 + e_1 T_1 + \ldots + e_n T_n \) where \( (T_0, T_1, \ldots, T_n) \) is an \((n+1)\)-tuple of linear commuting operators. The SC-functional calculus, which is defined for bounded but also for unbounded operators, associates to a suitable slice monogenic function \( f \) with values in the Clifford algebra \( \mathbb{R}_n \) the operator \( f(T) \). The \( \mathcal{F} \)-functional calculus has been defined, for bounded operators \( T \), by an integral transform. Such an integral transform comes from the Fueter’s mapping theorem and it associates to a suitable slice monogenic function \( f \) the operator \( \hat{f}(T) \), where \( \hat{f}(x) = \Delta^{\frac{n-1}{2}} f(x) \) and \( \Delta \) is the Laplace operator. Both functional calculi are based on the notion of \( \mathcal{F} \)-spectrum that plays the role that the classical spectrum plays for the Riesz-Dunford functional calculus. The aim of this paper is to study the bounded perturbations of the SC-resolvent operator and of the \( \mathcal{F} \)-resolvent operator. Moreover we will show some examples of equations that lead to the \( \mathcal{F} \)-spectrum.

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1. Introduction

The recent theory of slice monogenic functions, mainly developed in the papers [2], [3], [8], [9], [10], [12], turned out to be very important because of its applications to the so-called \( \mathcal{S} \)-functional calculus for \( n \)-tuples of non-necessarily commuting operators (bounded or unbounded), see [3] and [11]. The theory admits a quaternionic version of the \( \mathcal{S} \)-functional calculus for quaternionic linear operators which can be found in [4]. It is crucial to note that slice monogenic functions have a Cauchy formula with slice monogenic kernel that admits two expressions. These
two expressions of the Cauchy kernel are not equivalent when we want to define a functional calculus for non-necessarily commuting operators. In this paper, we will consider the case of commuting operators and the expression of the Cauchy kernel which gives rise to the definition of $\mathcal{F}$-spectrum which is the natural tool to treat the case of commuting operators.

Let $x = x_0 + e_1 x_1 + \ldots + e_n x_n$ and $s = s_0 + e_1 s_1 + \ldots + e_n s_n$ be paravectors in $\mathbb{R}^{n+1}$. We consider the Cauchy kernel written in the form

$$S_C^{-1}(s, x) = (s - \bar{x})(s^2 - 2\text{Re}[x]s + |x|^2)^{-1}$$

which is defined for $x^2 - 2xs_0 + |s|^2 \neq 0$. Let $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$, where $\mathbb{R}_n$ is the real Clifford algebra with $n$ imaginary units, $U$ is a suitable open set that contains the singularities of $S_C^{-1}(s, x)$. Let $I$ be a $1$-vector such that $I^2 = -1$ and let $C_I$ be the complex plane that contains 1 and $I$. Then we have the Cauchy formula

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} S_C^{-1}(s, x) ds_I f(s), \quad ds_I = -Ids, \quad (1.1)$$

where the integral does not depend on the open set $U$ and on the imaginary unit $I$. In the paper [5] we have introduced the so-called $SC$-functional calculus ($SC$ stands for slice-commuting), which is defined for bounded but also for unbounded commuting operators, starting from the above Cauchy formula

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} S_C^{-1}(s, T) ds_I f(s), \quad ds_I = -Ids. \quad (1.2)$$

The definition is well posed because the the integral in (1.2) does not depend on the open set $U$ and on the imaginary unit $I$. The $SC$-resolvent operator is defined by

$$S_C^{-1}(s, T) := (sI - \overline{T})(s^2I - s(T + \overline{T}) + TT)^{-1}$$

whose associated spectrum is the $\mathcal{F}$-spectrum of $T$ defined as:

$$\sigma_\mathcal{F}(T) = \{ s \in \mathbb{R}^{n+1} : s^2I - s(T + \overline{T}) + TT \text{ is not invertible} \}.$$ 

It is important to point out the meaning of the symbols. We have that, by definition, $T = T_0 + e_1 T_1 + \ldots + e_n T_n$, $\overline{T} = T_0 - e_1 T_1 - \ldots - e_n T_n$, so that $T + \overline{T} = 2T_0$, and since the components of $T$ commute, we have $T \overline{T} = T_0^2 + T_1^2 + \ldots + T_n^2$.

In the paper [6] we have proved the Fueter mapping theorem in integral form using the Cauchy formula (1.2). Precisely we have proved that, when $n$ is an odd number, given the slice monogenic function $f$, we can associate to it the monogenic function $\tilde{f}(x)$ by the integral transform

$$\tilde{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} \gamma_n(s - \overline{x})(s^2 - s(x + \overline{x}) + |x|^2)^{-\frac{n+1}{2}} ds_I f(s), \quad ds_I = -Ids, \quad (1.3)$$

where $\gamma_n$ is a given constant. We recall that the Fueter mapping theorem in differential form is given in [19] for $n$ odd and in Qian’s paper [16] in the general case. Later on, Fueter’s theorem has been generalized to the case in which a function $f$ as above is multiplied by a monogenic homogeneous polynomial of degree $k$, see
[15], [20] and to the case in which the function $f$ is defined on an open set $U$ not necessarily chosen in the upper complex plane, see [17].

We point out that formula (1.3) allows us to define the $\mathcal{F}$-functional calculus by replacing $x = x_0 + x_1 e_1 + \ldots + x_n e_n$ by $T = T_0 + T_1 e_1 + \ldots + T_n e_n$. More precisely, in [6] we have defined the following version of the monogenic functional calculus by setting

$$
\hat{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T) ds T f(s),
$$

(1.4)

where

$$
\mathcal{F}_n^{-1}(s, T) := \gamma_n(s I - T)(s^2 I - s(T + T) + T T)^{-\frac{n+1}{2}}.
$$

(1.5)

The functional calculus in (1.4) is well defined since the integral does not depend on the open set $U$ and on $I \in \mathbb{S}$. The natural notion of spectrum in this case is again the notion of $\mathcal{F}$-spectrum of $T$ as it is suggested by the definition of the $\mathcal{F}$-resolvent operator defined in (1.5).

The goal of this paper is to prove that bounded perturbations of the $\mathcal{F}$-resolvent operator and of the $\mathcal{F}$-resolvent operator produce bounded variations of the respective functional calculi.

We conclude by recalling that the well known theory of monogenic functions, see [1], [7], is the natural tool to define the monogenic functional calculus which has been well studied and developed by several authors, see the book of B. Jefferies [14] and the literature therein. For the analogies with the Riesz-Dunford functional calculus see for example the classical books [13] and [18].

2. Preliminary material

The setting in which we will work is the real Clifford algebra $\mathbb{R}_n$ over $n$ imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = i_1 \ldots i_r$, $i_\ell \in \{1, 2, \ldots, n\}$, $i_1 < \ldots < i_r$ is a multi-index, $e_A = e_{i_1} e_{i_2} \ldots e_{i_r}$ and $e_0 = 1$. In the Clifford algebra $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1, x_2, \ldots, x_n) \mapsto \underline{x} = x_1 e_1 + \ldots + x_n e_n$. An element $(x_0, x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j$ called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2$. The real part $x_0$ of $x$ will be also denoted by $\text{Re}[x]$. A function $f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ is seen as a function $f(x)$ of $x$ (and similarly for a function $f(\underline{x})$ of $\underline{x} \in U \subseteq \mathbb{R}^{n+1}$).

**Definition 2.1.** We will denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.,

$$
\mathbb{S} = \{ \underline{x} = e_1 x_1 + \ldots + e_n x_n : x_1^2 + \ldots + x_n^2 = 1 \}.
$$

Note that $\mathbb{S}$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$, while an element belonging
to \( \mathbb{C}_I \) will be denoted by \( u + Iv \), for \( u, v \in \mathbb{R} \). Observe that \( \mathbb{C}_I \), for every \( I \in \mathbb{S} \), is a 2-dimensional real subspace of \( \mathbb{R}^{n+1} \) isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.

Given a paravector \( x = x_0 + x \in \mathbb{R}^{n+1} \) let us set
\[
I_x = \begin{cases} \frac{x}{\|x\|} & \text{if } x \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}
\]

By definition we have that a paravector \( x \), with \( x \neq 0 \), belongs to \( \mathbb{C}_{I_x} \).

**Definition 2.2.** Given an element \( x \in \mathbb{R}^{n+1} \), we define \( [x] = \{ y \in \mathbb{R}^{n+1} : y = \text{Re}[x] + I|x| \} \), where \( I \in \mathbb{S} \).

**Remark 2.3.** The set \([x]\) is an \((n-1)\)-dimensional sphere in \( \mathbb{R}^{n+1} \). When \( x \in \mathbb{R} \), then \([x]\) contains \( x \) only. In this case, the \((n-1)\)-dimensional sphere has radius equal to zero.

**Definition 2.4 (Slice monogenic functions).** Let \( U \subseteq \mathbb{R}^{n+1} \) be an open set and let \( f : U \to \mathbb{R}_n \) be a real differentiable function. Let \( I \in \mathbb{S} \) and let \( f_I \) be the restriction of \( f \) to the complex plane \( \mathbb{C}_I \). We say that \( f \) is a (left) slice monogenic function, or \( s \)-monogenic function, if for every \( I \in \mathbb{S} \), we have
\[
\frac{1}{2} \left( \frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0.
\]
We denote by \( \mathcal{SM}(U) \) the set of \( s \)-monogenic functions on \( U \).

The natural class of domains in which we can develop the theory of \( s \)-monogenic functions are the so-called slice domains and axially symmetric domains.

**Definition 2.5 (Slice domains).** Let \( U \subseteq \mathbb{R}^{n+1} \) be a domain. We say that \( U \) is a slice domain (s-domain for short) if \( U \cap \mathbb{R} \) is non-empty and if \( \mathbb{C}_I \cap U \) is a domain in \( \mathbb{C}_I \) for all \( I \in \mathbb{S} \).

**Definition 2.6 (Axially symmetric domains).** Let \( U \subseteq \mathbb{R}^{n+1} \). We say that \( U \) is axially symmetric if, for every \( u + Iv \in U \), the whole \((n-1)\)-sphere \([u + Iv]\) is contained in \( U \).

Let us now introduce the notations necessary to deal with linear operators. By \( V \) and by \( V_n \) we denote a Banach space over \( \mathbb{R} \) with norm \( \| \cdot \| \) and \( V \otimes \mathbb{R}_n \), respectively. We recall that \( V_n \) is a two-sided Banach module over \( \mathbb{R}_n \) and its elements are of the type \( \sum_A v_A \otimes e_A \) (where \( A = i_1 \ldots i_r \), \( i_\ell \in \{1, 2, \ldots, n\}, i_1 < \ldots < i_r \) is a multi-index). The multiplications (right and left) of an element \( v \in V_n \) with a scalar \( a \in \mathbb{R}_n \) are defined as \( va = \sum_A v_A \otimes (ae_A) \) and \( av = \sum_A v_A \otimes (ae_A) \). For short, in the sequel we will write \( \sum_A v_A e_A \) instead of \( \sum_A v_A \otimes e_A \). Moreover, we define \( \|v\|_{V_n}^2 = \sum_A \|v_A\|_V^2 \).

Let \( B(V) \) be the space of bounded \( \mathbb{R} \)-homomorphisms of the Banach space \( V \) into itself endowed with the natural norm denoted by \( \| \cdot \|_{B(V)} \). If \( T_A \in B(V) \), we
can define the operator \( T = \sum_A T_A e_A \) and its action on \( v = \sum_B v_B e_B \) as \( T(v) = \sum_A B e_A e_B \). The set of all such bounded operators is denoted by \( B_n(V_n) \) and the norm is defined by \( \|T\|_{B_n(V_n)} = \sum_A \|T_A\|_{B(V)} \). Note that, in the sequel, we will omit the subscript \( B_n(V_n) \) in the norm of an operator and note also that \( \|TS\| \leq \|T\| \|S\| \). A bounded operator \( T = T_0 + \sum_{j=1}^n e_j T_j \), where \( T_\mu \in B(V) \) for \( \mu = 0, 1, \ldots, n \), will be called, with an abuse of notation, an operator in paravector form. The set of such operators will be denoted by \( B_{n,1}^0(V_n) \). The set of bounded operators of the type \( T = \sum_{j=1}^n e_j T_j \), where \( T_\mu \in B(V) \) for \( \mu = 1, \ldots, n \), will be denoted by \( \mathcal{B}_{n,1}^1(V_n) \) and \( T \) will be said operator in vector form. We will consider operators of the form \( T = T_0 + \sum_{j=1}^n e_j T_j \) where \( T_\mu \in B(V) \) for \( \mu = 0, 1, \ldots, n \) for the sake of generality, but when dealing with \( n \)-tuples of operators, we will embed them into \( B_n(V_n) \) as operators in vector form, by setting \( T_0 = 0 \). The subset of those operators in \( B_n(V_n) \) whose components commute among themselves will be denoted by \( \mathcal{B}_n(V_n) \). In the same spirit we denote by \( \mathcal{B}_n^0,1(V_n) \) the set of paravector operators with commuting components.

We now recall some definitions and results from [5], [6].

**Definition 2.7 (The \( \mathcal{F} \)-spectrum and the \( \mathcal{F} \)-resolvent sets).** Let \( T \in \mathcal{B}_n^0,1(V_n) \). We define the \( \mathcal{F} \)-spectrum of \( T \) as:

\[
\sigma_{\mathcal{F}}(T) = \{ s \in \mathbb{R}^{n+1} : s^2 I - s(T + \overline{T}) + TT^* \text{ is not invertible} \}.
\]

The \( \mathcal{F} \)-resolvent set of \( T \) is defined by

\[
\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).
\]

**Theorem 2.8 (Structure of the \( \mathcal{F} \)-spectrum).** Let \( T \in \mathcal{B}_n^0,1(V_n) \) and let \( p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R} \), such that \( p \in \sigma_{\mathcal{F}}(T) \). Then all the elements of the \((n-1)\)-sphere \([p_0 + p_1 I] \) belong to \( \sigma_{\mathcal{F}}(T) \). Thus the \( \mathcal{F} \)-spectrum consists of real points and/or \((n-1)\)-spheres.

**Theorem 2.9 (Compactness of \( \mathcal{F} \)-spectrum).** Let \( T \in \mathcal{B}_n^0,1(V_n) \). Then the \( \mathcal{F} \)-spectrum \( \sigma_{\mathcal{F}}(T) \) is a compact non-empty set. Moreover \( \sigma_{\mathcal{F}}(T) \) is contained in \( \{ s \in \mathbb{R}^{n+1} : |s| \leq \|T\| \} \).

**Definition 2.10.** Let \( T \in \mathcal{B}_n^0,1(V_n) \) and let \( U \subset \mathbb{R}^{n+1} \) be an axially symmetric \( s \)-domain containing the \( \mathcal{F} \)-spectrum \( \sigma_{\mathcal{F}}(T) \), and such that \( \partial(U \cap \mathbb{C}_I) \) is union of a finite number of continuously differential Jordan curves for every \( I \in \mathbb{S} \). Let \( W \) be an open set in \( \mathbb{R}^{n+1} \). A function \( f \in \mathcal{S}_M(W) \) is said to be locally \( s \)-monogenic on \( \sigma_{\mathcal{F}}(T) \) if there exists a domain \( U \subset \mathbb{R}^{n+1} \) as above such that \( \overline{U} \subset W \). We will denote by \( \mathcal{S}_{M_{\sigma_{\mathcal{F}}(T)}} \) the set of locally \( s \)-monogenic functions on \( \sigma_{\mathcal{F}}(T) \).

**Definition 2.11 (The \( \mathcal{S}\mathcal{C} \)-resolvent operator).** Let \( T \in \mathcal{B}_n^0,1(V_n) \) and \( s \in \rho_{\mathcal{F}}(T) \). We define the \( \mathcal{S}\mathcal{C} \)-resolvent operator as

\[
\mathcal{S}_{\mathcal{C}}^{-1}(s, T) := (sI - T)(s^2 I - s(T + \overline{T}) + TT^*)^{-1}.
\]
Definition 2.12 (The $SC$-functional calculus). Let $T \in BC_n^{0,1}(V_n)$ and $f \in SM_{\sigma_{\mathcal{F}}}(T)$. Let $U \subset \mathbb{R}^{n+1}$ be a domain as in Definition 2.10 and set $ds_I = ds/I$ for $I \in \mathbb{S}$. We define the $SC$-functional calculus as
\[
f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} S_{c}^{-1}(s, T) \, ds_I \ f(s). \tag{2.2}
\]

Definition 2.13 ($\mathcal{F}$-resolvent operator). Let $n$ be an odd number and let $T \in BC_n^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the $\mathcal{F}$-resolvent operator as
\[
\mathcal{F}_{n}^{-1}(s, T) := \gamma_n(s\mathcal{L} - \overline{T})(s^2\mathcal{L} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}},
\]
where
\[
\gamma_n := (-1)^{(n-1)/2}(n-1)/2(n-1)! \left(\frac{n-1}{2}\right)!.
\]

Next we define $\hat{f}(T)$ when $\hat{f}$ is a monogenic function which comes from an $s$-monogenic function $f$ via Fueter's theorem. The $\mathcal{F}$-functional calculus will be defined for those monogenic functions that are of the form $\hat{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$, where $f$ is an $s$-monogenic function. For the functional calculus associated to standard monogenic functions we mention the book [14].

Definition 2.14 (The $\mathcal{F}$-functional calculus). Let $n$ be an odd number and let $T \in BC_n^{0,1}(V_n)$. Let $U$ be an open set as in Definition 2.10. Suppose that $f \in SM_{\sigma_{\mathcal{F}}}(T)$ and let $\hat{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the $\mathcal{F}$-functional calculus as
\[
\hat{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} \mathcal{F}_{n}^{-1}(s, T) \, ds_I \ f(s). \tag{2.3}
\]

Remark 2.15. The definitions of the $SC$-functional calculus and of the $\mathcal{F}$-functional calculus are well posed since the integrals in (2.2) and in (2.3) are independent of $I \in \mathbb{S}$ and of the open set $U$.

3. Examples of equations for the $\mathcal{F}$-spectrum

Example (The case of Dirac operator). Let us consider the $n$-tuple of operators $(\partial_{x_1}, \ldots, \partial_{x_n})$, each of them acting on the vector space of functions of class $C^2$ over an open set $U \subset \mathbb{R}^{n+1}$. The vector operator associated to them is the Dirac operator
\[
T = \partial_{x_1} e_1 + \ldots + \partial_{x_n} e_n.
\]
Let us determine the equation which gives its $\mathcal{F}$-spectrum. We have $\overline{T} = -\partial_{x_1} e_1 - \ldots - \partial_{x_n} e_n$, and, since $\partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}$, for all $i, j = 1, \ldots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2 = \Delta$. The $\mathcal{F}$-spectrum is associated to the equation
\[
(s^2\mathcal{L} - s(T + \overline{T}) + T\overline{T})v = 0 \quad \text{for } v \neq 0
\]
which, in this case, becomes
\[
(s^2\mathcal{L} + \Delta)v = 0 \quad \text{for } v \neq 0. \tag{3.1}
\]
The paravector \( s \) can be considered as an element belonging to a complex plane \( s \in \mathbb{C}_{I_0} \), so we can assume that \( s = s_0 + s_1 I_0 \) is a solution of (3.1) for some \( I_0 \). Then the \( F \)-spectrum of \( T \) is given by

\[
\sigma_F(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.1)}} \{ s = s_0 + s_1 I, \text{ for all } I \in S \}.
\]

**Example (The case of second derivatives).** Let us consider the second-order operators \((\partial^2_{x_1}, \ldots, \partial^2_{x_n})\) each of them acting on the vector space of functions of class \( C^4 \) over an open set \( U \subseteq \mathbb{R}^{n+1} \), and let us write

\[
T = \partial^2_{x_1} e_1 + \ldots + \partial^2_{x_n} e_n.
\]

Determine the equation which gives its \( F \)-spectrum. We have \( T = -\partial^2_{x_1} e_1 - \ldots - \partial^2_{x_n} e_n \), and, since \( \partial^2_{x_i} x_j = \partial^2_{x_j} x_i \), for all \( i, j = 1, \ldots, n \) we also have \( T + T = 0 \) and \( T T = \sum_{j=1}^n A^{2j} \). The \( F \)-spectrum is associated to the equation

\[
(s^2 I + \partial^4_{x_1} + \ldots + \partial^4_{x_n})v = 0 \quad \text{for } v \neq 0.
\] (3.2)

We solve the equation (3.2) on the complex plane \( s \in \mathbb{C}_{I_0} \) for some \( I_0 \). Then the \( F \)-spectrum of \( T \) is given by

\[
\sigma_F(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.2)}} \{ s = s_0 + s_1 I, \text{ for all } I \in S \}.
\]

**Example (The case of powers of a real matrix).** Let \( A \) be a matrix \( n \times n \) with real entries and consider the operators \( T_j := A^j \) for \( j = 1, \ldots, n \). Determine the equation associated to the \( F \)-spectrum. It is well known that \( T_j T_k = T_k T_j \), for \( j, k = 1, \ldots, n \). So we consider the operator

\[
T = Ae_1 + \ldots + A^n e_n.
\]

We have \( T = -Ae_1 - \ldots - A^n e_n \), \( T + T = 0 \) and also \( T T = \sum_{j=1}^n A^{2j} \). The \( F \)-spectrum is associated to the equation

\[
(s^2 I + \sum_{j=1}^n A^{2j})v = 0 \quad \text{for } v \neq 0.
\] (3.3)

Let us conclude this short list of examples with an explicit computation of the \( F \)-spectrum.

**Example (The case of two triangular commuting matrices).** Let us consider \( a, b, \alpha, \beta \in \mathbb{R} \) and the two matrices:

\[
T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}.
\]

It is easy to verify that \( T_1 T_2 = T_2 T_1 \). So we associate to \( T_1 \) and \( T_2 \) the operator

\[
T = T_1 e_1 + T_2 e_2 = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix}.
\]
We have

\[
\mathbf{T} = \begin{bmatrix} \alpha e_1 + \alpha e_2 & \beta e_1 + \beta e_2 \\ 0 & \alpha e_1 + \beta e_2 \end{bmatrix},
\]

\[T + \mathbf{T} = 0\] and

\[
T\mathbf{T} = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}.
\]

The \(\mathcal{F}\)-spectrum is associated to the equation

\[
\left( s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & a^2 + \alpha^2 \end{bmatrix} \right) v = 0 \quad \text{for} \quad v \neq 0
\]

which becomes

\[
\begin{bmatrix} s^2 + a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & s^2 + a^2 + \alpha^2 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \quad \text{for} \quad \begin{bmatrix} u \\ w \end{bmatrix} \neq 0.
\]

Consider a paravector \(s\) on the complex plane \(\mathbb{C}_{I_0}\); with some calculation we obtain

\[s = \pm I_0 \sqrt{a^2 + \alpha^2}.\]

Thus the \(\mathcal{F}\)-spectrum of \(T\) is given by

\[
\sigma_{\mathcal{F}}(T) = \{ \pm I \sqrt{a^2 + \alpha^2} \text{ for all } I \in \mathbb{S} \}.
\]

4. Bounded perturbations of the \(\mathcal{S}\mathcal{C}\)-resolvent

**Lemma 4.1.** The set \(\mathcal{U}(V_n)\) of elements in \(\mathcal{B}_n(V_n)\) which have inverse in \(\mathcal{B}_n(V_n)\) is an open set in the uniform topology of \(\mathcal{B}_n(V_n)\). If \(\mathcal{U}(V_n)\) contains an element \(A\), then it contains the ball

\[
\Sigma = \{ B \in \mathcal{B}_n(V_n) : \| A - B \| < \| A^{-1} \|^{-1} \}.
\]

If \(B \in \Sigma\), its inverse is given by the series

\[
B^{-1} = A^{-1} \sum_{m \geq 0} [(A - B)A^{-1}]^m. \tag{4.1}
\]

Furthermore, the map \(A \mapsto A^{-1}\) from \(\mathcal{U}(V_n)\) onto \(\mathcal{U}(V_n)\) is a homeomorphism in the uniform operator topology.

**Proof.** See Lemma 7.1 in [3]. \(\square\)

In order to state our results, we need the following definitions:

**Definition 4.2.** Let \(T \in \mathcal{B}_{n,0,1}(V_n)\). We denote by \(\sigma_L(T)\) the so-called left spectrum of \(T\) related to the resolvent operator \((s\mathcal{I} - T)^{-1}\) that is defined as

\[
\sigma_L(T) = \{ s \in \mathbb{R}^{n+1} : s\mathcal{I} - T \quad \text{is not invertible in} \quad \mathcal{B}_{n,0,1}(V_n) \},
\]

where the notation \(s\mathcal{I}\) in \(B^R(V)\) means that \((s\mathcal{I})(v) = sv\).

**Definition 4.3.** Let \(\mathcal{W}\) be a subset of \(\mathbb{R}^{n+1}\). We denote by \(B(\mathcal{W}, \varepsilon)\), for \(\varepsilon > 0\), the \(\varepsilon\)-neighborhood of \(\mathcal{W}\) defined as

\[
B(\mathcal{W}, \varepsilon) := \{ x \in \mathbb{R}^{n+1} : \inf_{s \in \mathcal{W}} |s - x| < \varepsilon \}. 
\]

Lemma 4.4. Let $T, Z \in \mathcal{B}C_0^{0,1}(V_n)$, $s \notin \sigma_L(T) \cup \sigma_L(Z)$ and consider

\[
S_C(s, T) = sI - (sI - T)(sI - T)^{-1}, \quad (4.2)
\]

\[
S_C(s, Z) = sI - (sI - Z)(sI - Z)^{-1}. \quad (4.3)
\]

Then there exists a strictly positive constant $K(s)$, depending on $s$ and also on the operators $T$ and $Z$, such that

\[
\|S_C(s, T) - S_C(s, Z)\| \leq K(s)\|T - Z\|. \quad (4.4)
\]

Proof. Consider the chain of equalities

\[
S_C(s, T) - S_C(s, Z) = (sI - Z)Z(sI - Z)^{-1} - (sI - T)T(sI - T)^{-1}
\]

\[
= (sI - Z)Z(sI - Z)^{-1} - (sI - T)Z(sI - Z)^{-1}
\]

\[
+ (sI - T)Z(sI - Z)^{-1} - (sI - T)T(sI - T)^{-1}
\]

\[
= (T - Z)Z(sI - Z)^{-1} + (sI - T)[Z(sI - Z)^{-1} - T(sI - T)^{-1}]
\]

\[
= (T - Z)Z(sI - Z)^{-1} + (sI - T)[(Z - T)(sI - Z)^{-1} + T((sI - Z)^{-1} - (sI - T)^{-1})]
\]

\[
= (T - Z)Z(sI - Z)^{-1} + (sI - T)[(Z - T)(sI - Z)^{-1} + T(sI - Z)^{-1}(Z - T)(sI - T)^{-1}].
\]

By taking the norm and observing that $\|T - Z\| = \|T - Z\|$, we have

\[
\|S_C(s, T) - S_C(s, Z)\| \leq \|T - Z\| \left(\|Z\| \|(sI - Z)^{-1}\| + \|sI - T\| \|(sI - Z)^{-1}\| \|(sI - T)^{-1}\|\right).
\]

If we now set

\[
K(s) := \|(sI - Z)^{-1}\| \left(\|Z\| + \|sI - T\| \left[1 + \|T\| \|(sI - T)^{-1}\|\right]\right), \quad (4.5)
\]

we have that $K(s) > 0$ and we get the statement. \hfill \Box

Lemma 4.5. Let $T, Z \in \mathcal{B}C_0^{0,1}(V_n)$, $s \in \rho_F(T)$, $s \notin \sigma_L(T) \cup \sigma_L(Z)$ and suppose that

\[
\|T - Z\| < \frac{1}{K(s)}\|S_C^{-1}(s, T)\|^{-1},
\]

where $K(s)$ is defined in (4.5). Then $s \in \rho_F(Z)$ and

\[
S_C^{-1}(s, Z) - S_C^{-1}(s, T) = S_C^{-1}(s, T) \sum_{m \geq 1} [(S_C(s, T) - S_C(s, Z))S_C^{-1}(s, T)]^m.
\]

(4.6)
Proof. Let us recall (4.2) and (4.3) and set

\[ A := S_C(s, T), \quad B := S_C(s, Z), \quad A^{-1} = S_C^{-1}(s, T). \]  

(4.7)

By formula (4.1) in Lemma 4.1 with \( B^{-1} := S_C^{-1}(s, Z) \), we get

\[ S_C^{-1}(s, Z) = S_C^{-1}(s, T) \sum_{m \geq 0} [(S_C(s, T) - S_C(s, Z))S_C^{-1}(s, T)]^m. \]  

(4.8)

The series in (4.8) converges since

\[ \|(S_C(s, T) - S_C(s, Z))S_C^{-1}(s, T)\| \leq K(s)\|T - Z\|\|S_C^{-1}(s, T)\| < 1, \]

so we get the statement. \( \square \)

**Theorem 4.6.** Let \( T, Z \in BC_n^0(V_n), s \in \rho_F(T), s \not\in \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z}). \) Let \( \varepsilon > 0 \) and consider the \( \varepsilon \)-neighborhood \( B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \) of \( \sigma_F(T) \cup \sigma_L(\overline{T}) \). Then there exists \( \delta > 0 \) such that, for \( \|T - Z\| < \delta \), we have

\[ \sigma_F(Z) \subseteq B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \]

and

\[ \|S_C^{-1}(s, Z) - S_C^{-1}(s, T)\| < \varepsilon, \text{ for } s \not\in B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon). \]

Proof. Let \( \overline{T}, \overline{Z} \in BC_n^0(V_n) \) and let \( \varepsilon > 0 \). Thanks to Lemma 4.1 there exists a \( \eta > 0 \) such that if

\[ \|T - Z\| < \eta \]

then \( \sigma_L(\overline{Z}) \subseteq B(\sigma_L(\overline{T}), \varepsilon) \), where \( B(\sigma_L(\overline{T}), \varepsilon) \) is the \( \varepsilon \)-neighborhood of \( \sigma_L(\overline{T}) \). So we can always choose \( \eta \) such that \( \sigma_L(\overline{Z}) \subseteq B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \). Consider the function \( K(s) \) defined in (4.5) and observe that the constant \( K_\varepsilon \) defined by

\[ K_\varepsilon = \sup_{s \not\in B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon)} K(s) \]

is finite since \( s \not\in B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \), for the above observation \( \sigma_L(\overline{Z}) \subseteq B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \) and because

\[ \lim_{s \to \infty} \|(sI - \overline{Z})^{-1}\| = \lim_{s \to \infty} \|(sI - \overline{T})^{-1}\| = 0. \]

Observe that since \( s \in \rho_F(T) \) the map \( s \mapsto \|S_C^{-1}(s, T)\| \) is continuous and

\[ \lim_{s \to \infty} \|S_C^{-1}(s, T)\| = 0, \]

for \( s \) in the complement set of \( B(\sigma_F(T) \cup \sigma_L(\overline{T}), \varepsilon) \) we have that there exists a positive constant \( N_\varepsilon \) such that

\[ \|S_C^{-1}(s, T)\| \leq N_\varepsilon. \]

From Lemma 4.5, if \( \delta_1 > 0 \) is such that \( \|Z - T\| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1 \), then \( s \in \rho_F(Z) \) and
Proof. We recall that operator $U$ is

$$\|S_c^{-1}(s, Z) - S_c^{-1}(s, T)\| \leq \frac{\|S_c^{-1}(s, T)\|^2 \|S_c(s, T) - S_c(s, Z)\|}{1 - \|S_c^{-1}(s, T)\| \|S_c(s, T) - S_c(s, Z)\|}$$

$$\leq \frac{N_2^2K_\varepsilon\|Z - T\|}{1 - N_2K_\varepsilon\|Z - T\|} < \varepsilon$$

if we take

$$\|Z - T\| < \delta_2 := \frac{\varepsilon}{K_\varepsilon(N_2^2 + \varepsilon N_2)}.$$ 

To get the statement it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$. \qed

**Theorem 4.7.** Let $T, Z \in B^{\sigma_f}_{n^2}(V_n)$, $f \in SM_{\sigma_f}(T)$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in SM_{\sigma_f}(Z)$ and

$$\|f(Z) - f(T)\| < \varepsilon.$$ 

**Proof.** We recall that operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} S_c^{-1}(s, T) \ ds_I \ f(s)$$

where $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in S$. Suppose that $U$ is an $\varepsilon$-neighborhood of $\sigma_f(T) \cup \sigma_L(T)$ and it is contained in the domain in which $f$ is $s$-monogenic. By Lemma 4.6 there is a $\delta_1 > 0$ such that $\sigma_f(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently $f \in SM_{\sigma_f}(Z)$ for $\|Z - T\| < \delta_1$. By Lemma 4.6 $S_c^{-1}(s, T)$ is uniformly near to $S_c^{-1}(s, Z)$ with respect to $s \in \partial(U \cap C_I)$ for $I \in S$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|f(T) - f(Z)\| = \frac{1}{2\pi} \||S_c^{-1}(s, T) - S_c^{-1}(s, Z)\| \int_{\partial(U \cap C_I)} ds_I \ f(s)\| < \varepsilon.$$ \qed

5. **Bounded perturbations of the $F$-resolvent**

Let $n$ be an odd number. For $s \in \rho_f(T)$ the $F$-resolvent operator associated to $T$ is

$$F_n^{-1}(s, T) := \gamma_n(sI - \overline{T})(s^2I - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}, \quad (5.1)$$

while its inverse is

$$F_n(s, T) := \frac{1}{\gamma_n}(s^2I - s(T + \overline{T}) + T\overline{T})^{\frac{n+1}{2}}(sI - \overline{T})^{-1}, \quad (5.2)$$

for $s \notin \sigma_L(\overline{T})$. Analogously for $s \in \rho_f(Z)$ the $F$-resolvent operator associated to $Z$ is

$$F_n^{-1}(s, Z) := \gamma_n(sI - \overline{Z})(s^2I - s(Z + \overline{Z}) + Z\overline{Z})^{-\frac{n+1}{2}}, \quad (5.3)$$

and it has the inverse

$$F_n(s, Z) := \frac{1}{\gamma_n}(s^2I - s(Z + \overline{Z}) + Z\overline{Z})^{\frac{n+1}{2}}(sI - \overline{Z})^{-1}, \quad (5.4)$$

for $s \notin \sigma_L(\overline{Z})$. 

Now observe that

\[ \| F_n(s, T) - F_n(s, Z) \| \leq C_n(s) (\|s\| + \vartheta)^n \| T - Z \|, \]  

(5.5)

where \( \vartheta := \max\{\|T\|, \|Z\|\} \).

**Proof.** For simplicity let us set the positions \( \frac{n+1}{2} := k + 1 \), for \( k \in \mathbb{N} \), so that \( k = \frac{n-1}{2} \), for \( n = 1, 3, 5, \ldots \). The case \( k = 0 \) has been studied in the previous section. Here we consider \( k \geq 1 \). We set \( \beta_k := \gamma_{2k+1} \) and we define, for \( s \in \rho_F(T) \),

\[ \tilde{F}_k(s, T) := \beta_k S_C^{-1}(s, T) (s^2 I - s(T + \overline{T}) + TT)^{-k}. \]  

(5.6)

The inverse of operator \( \tilde{F}_k^{-1}(s, T) \) exists for \( s \not\in \sigma_L(\overline{T}) \) and is given by

\[ \tilde{F}_k(s, T) = \frac{1}{\beta_k} (s^2 I - s(T + \overline{T}) + TT)^k S_C(s, T), \]  

(5.7)

while the inverse of operator \( \tilde{F}_k^{-1}(s, Z) \) exists for \( s \not\in \sigma_L(\overline{Z}) \) and is given by

\[ \tilde{F}_k(s, Z) = \frac{1}{\beta_k} (s^2 I - s(Z + \overline{Z}) + ZZ)^k S_C(s, Z). \]  

(5.8)

Consider (5.7) and (5.8) for \( k = 1 \); we have

\[
\begin{align*}
\beta_1 [\tilde{F}_1(s, T) - \tilde{F}_1(s, Z)] &= (s^2 I - s(T + \overline{T}) + TT) S_C(s, T) - (s^2 I - s(Z + \overline{Z}) + ZZ) S_C(s, Z) \\
&= (s^2 I - s(T + \overline{T}) + TT) S_C(s, T) - (s^2 I - s(T + \overline{T}) + TT) S_C(s, Z) \\
&\quad + (s^2 I - s(T + \overline{T}) + TT) S_C(s, Z) - (s^2 I - s(Z + \overline{Z}) + ZZ) S_C(s, Z) \\
&= (s^2 I - s(T + \overline{T}) + TT) [S_C(s, T) - S_C(s, Z)] \\
&\quad + [-s(T + \overline{T}) + TT + s(Z + \overline{Z}) - ZZ]) S_C(s, Z) \\
&= (s^2 I - s(T + \overline{T}) + TT) [S_C(s, T) - S_C(s, Z)] \\
&\quad + [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(T - \overline{Z})] S_C(s, Z)
\end{align*}
\]

and taking the norm we get

\[
\begin{align*}
|\beta_1| \|\tilde{F}_1(s, T) - \tilde{F}_1(s, Z)\| &\leq (|s|^2 + 2|s| \|T\| + \|TT\|) \|S_C(s, T) - S_C(s, Z)\| \\
&\quad + [2|s| \|Z - T\| + \|T - Z\| (\|T\| + \|Z\|)] \|S_C(s, Z)\| \\
&\leq (|s| + \vartheta)^2 \|S_C(s, T) - S_C(s, Z)\| + [2(|s| + \vartheta) \|Z - T\|] \|S_C(s, Z)\|.
\end{align*}
\]

Now observe that

\[ (|s| + \vartheta)^{-1} \|S_C(s, Z)\| \leq (|s| + \vartheta)^{-1} [\|s\| + \|(sI - \overline{Z})\| \|Z\| \|(sI - \overline{Z})^{-1}\|] =: M(s) \]  

(5.9)
where \( M(s) \) is a continuous function since \( s \not\in \sigma_L(Z) \). Using Lemma 4.4 we get
\[
\| \tilde{F}_1(s, T) - \tilde{F}_1(s, Z) \| \leq \frac{1}{|\beta_1|} [K(s) + 2M(s)]|[|s| + \vartheta]^2\|Z - T\|. \tag{5.10}
\]
We now use the induction principle. We assume that the estimate
\[
\| \tilde{F}_k(s, T) - \tilde{F}_k(s, Z) \| \leq \frac{1}{|\beta_k|} ([|s| + \vartheta]^{2k}) [K(s) + 2kM(s)]\|Z - T\|. \tag{5.11}
\]
holds for \( k \geq 1 \). Observe that (5.8) implies that the estimate
\[
\| \mathcal{F}_k(s, Z) \| \leq \frac{1}{|\beta_k|} ([|s| + \vartheta]^{2k}) \|S_C(s, Z)\|. \tag{5.12}
\]
holds. We prove that
\[
\| \tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z) \| \leq \frac{1}{|\beta_{k+1}|} ([|s| + \vartheta]^{2(k+1)}) [K(s) + 2(k+1)M(s)]\|Z - T\|.
\]
In fact, we have that
\[
\beta_{k+1}(\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z))
= \beta_k(s^2I - s(T + \overline{T}) + T\overline{T}) \tilde{F}_k(s, T) - \beta_k(s^2I - s(Z + \overline{Z}) + Z\overline{Z}) \tilde{F}_k(s, Z)
= \beta_k(s^2I - s(T + \overline{T}) + T\overline{T}) [\tilde{F}_k(s, T) - \tilde{F}_k(s, Z)]
- \beta_k[s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - Z)] \tilde{F}_k(s, Z)
\]
and taking the norms we have
\[
|\beta_{k+1}|\|\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)\|
\leq |\beta_k|[|s| + \vartheta]^2\|\tilde{F}_k(s, T) - \tilde{F}_k(s, Z)\| + 2\beta_k(|s| + \vartheta) \|\mathcal{F}_k(s, Z)\|\|Z - T\|.
\]
Using (5.11) and (5.12) we obtain
\[
|\beta_{k+1}|\|\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)\|
\leq (|s| + \vartheta)^{2k+2}[K(s) + 2kM(s)]\|Z - T\| + 2(|s| + \vartheta)^{2k+1} \|S_C(s, Z)\|\|Z - T\|
\leq (|s| + \vartheta)^{2k+2}[K(s) + 2kM(s) + 2(|s| + \vartheta)^{-1} \|S_C(s, Z)\|]\|Z - T\|
\leq (|s| + \vartheta)^{2k+2}[K(s) + 2(k+1)M(s)]\|Z - T\|.
\]
Setting \( \tilde{C}_k(s) := \frac{1}{|\beta_k|}[K(s) + 2kM(s)] \) the constant \( C_n(s) \) in estimate (5.5) is given by
\[
C_n(s) := \frac{1}{|\gamma_n|}[K(s) + (n - 1)M(s)]. \tag{5.13}
\]
This concludes the proof.

\[\square\]

**Lemma 5.2.** Let \( n \) be an odd number, \( T, Z \in \mathcal{BC}_n^{0,1}(V_n) \), let \( s \in \rho_F(T) \), \( s \not\in \sigma_L(T) \cup \sigma_L(Z) \) and suppose that
\[
\|T - Z\| < \frac{1}{C_n(s)}(|s| + \vartheta)^{-(n-1)}\|\mathcal{F}_n^{-1}(s, T)\|^{-1},
\]
where $C_n(s)$ is defined in (5.13). Then $s \in \rho_F(Z)$ and
\[
F_n^{-1}(s, Z) - F_n^{-1}(s, T) = F_n^{-1}(s, T) \sum_{m \geq 1} [(F_n(s, T) - F_n(s, Z))F_n^{-1}(s, T)]^m. \tag{5.14}
\]

**Proof.** Let us recall (5.2), (5.4) and set
\[
A := F_n(s, T), \quad B := F_n(s, Z), \quad A^{-1} = F_n^{-1}(s, T)(s, T). \tag{5.15}
\]
By Lemma 4.1, formula (4.1), for $B^{-1} := F_n^{-1}(s, Z)$ we get
\[
F_n^{-1}(s, Z) = F_n^{-1}(s, T) \sum_{m \geq 0} [(F_n(s, T) - F_n(s, Z))F_n^{-1}(s, T)]^m. \tag{5.16}
\]
Using the hypothesis, we have that the series converges since
\[
\|((F_n(s, T) - F_n(s, Z))F_n^{-1}(s, T))\| \\
\leq \|((F_n(s, T) - F_n(s, Z))\| \|F_n^{-1}(s, T)\| \\
\leq C_n(s)(|s| + \vartheta)^{n-1} \|Z - T\| \|F_n^{-1}(s, T)\| < 1. \tag*{□}
\]

**Theorem 5.3.** Let $n$ be an odd number, $T, Z \in BC_n^0(V_n)$, $s \in \rho_F(T), s \not\in \sigma_L(T), \sigma_L(Z)$. Let $\varepsilon > 0$ and consider the $\varepsilon$-neighborhood $B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)$ of $\sigma_F(T) \cup \sigma_L(T)$. Then there exists $\delta > 0$ such that, for $\|T - Z\| < \delta$, we have
\[
\sigma_F(Z) \subseteq B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)
\]
and
\[
\|F_n^{-1}(s, Z) - F_n^{-1}(s, T)\| < \varepsilon, \quad \text{for} \quad s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon).
\]

**Proof.** Let $T, Z \in BC_n^0(V_n)$ and let $\varepsilon > 0$. Thanks to Lemma 4.1 there exists a $\eta > 0$ such that if
\[
\|T - Z\| < \eta,
\]
then $\sigma_L(Z) \subset B(\sigma_L(T), \varepsilon)$, where $B(\sigma_L(T), \varepsilon)$ is the $\varepsilon$-neighborhood of $\sigma_L(T)$. So we can always choose $\eta$ such that $\sigma_L(Z) \subset B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)$. Consider the function $C_n(s)$ defined in (5.13). The constant $C_{n, \varepsilon}$ defined as
\[
C_{n, \varepsilon} = \sup_{s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)} C_n(s)
\]
is finite because $s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)$ and
\[
\lim_{s \to \infty} \|sI - Z\|^{-1} = \lim_{s \to \infty} \|sI - T\|^{-1} = 0.
\]
Observe that the since $s \in \rho_F(T)$ map $s \mapsto \|F_n^{-1}(s, T)\|$ is continuous and
\[
\lim_{s \to \infty} \|F_n^{-1}(s, T)\| = 0,
\]
and so for $s$ in the complement set of $B(\sigma_F(T) \cup \sigma_L(T, \varepsilon)$ we have that there exists a positive constant $M_\varepsilon$ such that
\[
\|F_n^{-1}(s, T)\| \leq M_\varepsilon.
\]
From Lemma 5.2 if $\delta_1 > 0$ is such that
\[
\|Z - T\| < \frac{1}{C_{n,\varepsilon}M_\varepsilon} := \delta_3,
\]
then $s \in \rho_{F}(Z)$ and
\[
\|F^{-1}_n(s, Z) - F^{-1}_n(s, T)\|
\leq \frac{\|F^{-1}_n(s, T)\|^2 \|F_n(s, T) - F(s, Z)\|}{1 - \|F^{-1}_n(s, T)\| \|F_n(s, T) - F(s, Z)\|}
\leq \frac{M^2 C_{n,\varepsilon} \|Z - T\|}{1 - M_\varepsilon C_{n,\varepsilon} \|Z - T\|} < \varepsilon
\]
if we take
\[
\|Z - T\| < \delta_4 := \frac{\varepsilon}{C_{n,\varepsilon}(M^2\varepsilon + \varepsilon M_\varepsilon)}.
\]
To get the statement it suffices to set $\delta = \min\{\eta, \delta_3, \delta_4\}$.

**Theorem 5.4.** Let $n$ be an odd number, $T, Z \in BC_{n}^{0,1}(V_n)$, $f \in SM_{\sigma_{F}(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in SM_{\sigma_{F}(Z)}$ and
\[
\|\tilde{f}(Z) - \tilde{f}(T)\| < \varepsilon.
\]

**Proof.** We recall that
\[
\tilde{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} F^{-1}_n(s, T) \, ds_I \, f(s)
\]
and $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Let $U$ be an $\varepsilon$-neighborhood of $\sigma_{F}(T) \cup \sigma_{L}(T)$ contained in the domain in which $f$ is $s$-monogenic. By Lemma 5.3 there is a $\delta_1 > 0$ such that $\sigma_{F}(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently, $f \in SM_{\sigma_{F}(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 5.3, $F^{-1}_n(s, T)$ is uniformly near to $F^{-1}_n(s, Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get
\[
\|\tilde{f}(T) - \tilde{f}(Z)\| = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} [F^{-1}_n(s, T) - F^{-1}_n(s, Z)] \, ds_I \, f(s)\| < \varepsilon.
\]

**References**


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