Chapter 2
Computation in Complex Systems

Complex systems science is the study of large collections of (generally simple) entities, where the global behaviour is a non-trivial result of the local interactions of the individual elements [1]. This approach seeks a fundamental understanding of how such collective behaviour results from these interactions between simple individuals. In particular, it seeks to gain and apply this understanding across many different disciplines, examining both natural and man-made systems as apparently diverse as insect colonies, the brain, the immune system, economies and the world wide web [2]. Complex behaviour is often described as incorporating elements of both order and disorder (or chaos), and these elements can be seen in all of the above, e.g. path-following (order) versus exploration (disorder) in ant foraging.

While no common framework has been established for analysis of time-series of dynamics in complex systems science [1], increasingly "the notion of computation is being imported to explain the behaviour of" these complex systems [2]. To the lay reader, the concept of computation may seem rather distinct from these fields, yet the application is well-reasoned since the interactions between individuals can be seen as communication or signalling, and the system as a whole as processing information in determining its collective behaviour via a distributed computation.

The most prominent example of the use of the notion of computation to explain complex behaviour is with respect to cellular automata (CAs). CAs are discrete dynamical systems which are known to support complex computation, and have been used to model complex systems in nature. Their computation is qualitatively understood in terms of emergent coherent structures which are widely accepted to embody information storage, transfer and modification. Indeed this understanding is used to explain the computational function of similar emergent structures in neural circuits [3] and in plants [4]. Crucially however, quantitative evidence for such understanding is notably absent.

A quantitative understanding of the dynamics of these operations on information could provide a common analytic framework for complex systems, and would have important ramifications for the whole field. For example, insights into network topology in nature (e.g. [5, 6]) have been some of the greatest contributions of complex

J. T. Lizier, *The Local Information Dynamics of Distributed Computation in Complex Systems*, Springer Theses, DOI: 10.1007/978-3-642-32952-4_2,

systems science, however it is widely acknowledged [2, 7–9] that understanding the *dynamics* of networks is the "next frontier" [8]. In particular, "the main challenge is understanding the dynamics of the propagation of information…in networks, and how these networks process such information" [9]. Another prominent example are attempts to guide the design of self-organised systems using an understanding of how they structure information [10]. Here we suggest that the task under design can be considered as a distributed computation, with an opportunity then for a framework for information dynamics to provide quantitative insights to such design.

Information theory provides the logical platform for our investigations, since information is the language of computation. Indeed, information theory has proven successful in analysing complex systems [1], initially through studies of characterising order-disorder continua as well as measuring complexity. It has many important features (e.g. being mathematically abstract) that give it the potential to become a leading framework for analysis and design of complex systems. Yet what we consider to be the most important of these features, the ability to analyse the information dynamics of distributed computation, is not yet properly established.

This chapter is used to introduce the reader to the two foundations of our work, complex systems science in Sect. 2.1 and information theory in Sect. 2.2. We will introduce the basic information-theoretic concepts used here, in particular the approach to studying information dynamics on a local scale in space and time. The chapter is also used to strongly highlight the need for quantitative insights into the information dynamics of computation in complex systems, and to introduce several relevant models that are analysed in later chapters. In Sect. 2.3 we describe the current state of understanding of distributed computation in cellular automata, the most important domain for theoretical discussions of this concept. We introduce the reader to network science in Sect. 2.4, highlighting the opportunities for a theory of information dynamics to complement this understanding of static structure. As an illustration, we describe an important model used in later analysis: random Boolean networks (RBNs). Finally, we discuss the use of complex systems science and information theory to design self-organised systems in Sect. 2.5. These examples make the opportunity to contribute a *framework for the information dynamics of distributed computation in complex systems* abundantly clear.

2.1 Complex Systems

Complex systems science is concerned with the study of systems which have two key characteristics [1]:

- They are composed of many simple elements;
- The elements interact in a non-trivial fashion.

Non-triviality of the interaction is particularly important to this definition. Prokopenko et al. [1] state that systems with a huge number of components interacting *trivially* are the domain of statistical mechanics, while those with *precisely defined* and

constrained interactions are the domains of fields such as chemistry and engineering. Complex systems science however is concerned with the overlap of these, where non-trivial interaction of a large number of elements leads to intricate non-linear dynamics, violating classical (i.e. linear) assumptions. Complex systems science is important because of the prevalence of these characteristics in natural and man-made systems, and the challenge they present to traditional linear analysis.

Typically, computer simulation is used as the primary tool for the study of such systems. Here *agent-based modelling* [11, 12] considers the elements of the system as black boxes whose behavioural rules we know but whose internal structure we do not care about. The global behaviour of the system is allowed to emerge from the interaction of these simulated agents, modelling such emergence in the real-world system. Agent-based modelling has been described as "the most important conceptual tool introduced by complexity science" [11].

In one sense, the title *complex* captures the *difficult* nature of analysis of these systems. In another sense, it also captures the essence of their *interesting* nature. This is because the interaction of the elements on a microscopic level can give rise to sophisticated organisation of the system at a macroscopic level. Put another way, they can exhibit global behaviour which is interesting but not an obvious consequence of the local interactions. This is referred to as *emergent behaviour* [2].

Some authors go further in their definitions for complex systems. For example, Mitchell [2] proposes a complex system to be: "a system in which large numbers of components with no central control and simple rules of operation give rise to complex collective behaviour, sophisticated information processing, and adaptation via learning or evolution." There is no question that these terms are an essential part of our understanding of complex systems, but the extent to which some of these terms (e.g. adaptation) are necessarily part of complex systems or complex behaviour is debatable [1]. In the next sections, we discuss some of these related concepts. Subsequently, we consider the motivation for studying complex systems in more depth.

2.1.1 Order, Disorder and Phase Transitions

The underlying nature of complex systems in non-linear dynamics provides a very strong link to chaos theory (e.g. see [13]); indeed this field can be seen as one of the forerunners to complex systems science.[1] Complex systems theory is subtly different from chaos theory however: while chaos theory focuses on apparent randomness arising from very simple systems, complex systems science focuses on emergent global behaviour or organisation from non-trivial distributed interactions.

Importantly, complex systems are typically described as combining elements of order and randomness[2] to create truly intricate behaviour [11]. For example,

[1] For introductions to the beginnings of complex systems science, see [2, 11, 12, 14].

[2] In the sense of chaotic behaviour.

economies involve regulation and a perception of rational behaviour at the same time as wide variation in individual behaviours and unforeseen fluctuations in market dynamics.

Indeed, truly complex behaviour is neither completely ordered, nor completely random. Ordered systems are perfectly structured and therefore simple to predict [11]. Completely disordered systems cannot be predicted at all on an individual level, but prediction of average behaviour is not only possible but trivial. Complex systems on the other hand embody a *duality* between dependence and independence of their components, making prediction of them possible but non-trivial. Such observations are generalised by some authors in suggesting that complex behaviour occurs at a *phase transition* between ordered and chaotic behaviour (generalised in the *edge of chaos* hypothesis [15, 16]). Certainly there are many order-chaos phase transitions (in particular when they are dictated by a single order-chaos parameter) where complexity has been measured in some way to be maximised in the transition, e.g. maximisation of co-operative behaviour among ants at intermediate levels of ant density [17]. However, suggestions that such transition properties are universal, or that complex computation only occurs at such transitions in all systems[3] are strongly criticised in [18, 19].

2.1.2 Self-Organisation

The intricate link between the concepts of *self-organisation* and complex systems is shown by the phrasing "no central control" in Mitchell's definition [2] above.

A system is considered to be self-organised where it demonstrates two key features (see [20–22]):

- An increase in organisation (structure and/or functionality);
- Dynamics not guided by any centralised or external control agent.

These principles are generally accepted, although there remains some contention about their specific details. This includes how to quantify organisation (which differs between [20–22]), as well as the nature of external control and the extent to which it could be permitted.

Regardless of these specifics, it is clear that the global organisation in such systems result from distributed and localised interaction between the elements of the system; for example, in a school of fish individuals moderate their movement with reference to immediate neighbours rather than that of a central fish or of the whole school [23]. As such, self-organisation is a popular focus within complex systems science.

Related here is the concept *self-organised criticality*, where a system self-organises to a critical state near an order-chaos phase transition, and is stable to perturbations away from this state [24, 25]. *Scale-free* phenomena (or *fractal* or

[3] Particularly those with less well-defined or more complex transitions.

self-similar structure), where properties of agents or event sizes or inter-event distribution times follow a power-law, are a signature of these critical states.

2.1.3 Motivation for Studying Complex Systems

Motivations for studying complex and self-organised systems can be somewhat divided between *science*, or attempts to understand such systems, and *engineering*, or attempts to design or manipulate such systems for our own benefit.

Developing an understanding of complex systems is of immense importance, because such systems are widely distributed throughout nature. For example, Léveillé et al. state that understanding how "neurons cooperate to control behavioural processes is a fundamental problem in computational neuroscience" [26]. Other prominent examples include swarm behaviour [27], ant foraging [28], and heart beats [29]. Furthermore, complex behaviour arises in man-made systems, e.g. the internet [30] and city size distributions [25]. Complex systems science is very much an interdisciplinary field, both in terms of application domains and analytical approaches.

There are many open questions which require addressing before we can properly claim to have full understanding of the dynamics of these systems, including how such systems arise and how to classify types of systems and behaviour. Of prime importance is the common acceptance of how to quantify the concept of complexity, and sub-concepts such as self-organisation and emergence. Notions of computation are increasingly being used to describe complex systems; we will consider this perspective in relation to cellular automata in Sect. 2.3 (once we have established the required information-theoretical background). Also, fully understanding the dynamics of evolution (e.g. see [31]) is particularly important, because complex self-organised systems in nature have typically evolved over millions of years to address problems that are simple to state but hard to solve [28].

As such, engineering can and should take inspiration from these systems (e.g. as in [32]). This is particularly true because traditional engineering designs are finding their limitations in many situations: they are centralist, subject to single points of failure, have low tolerance to errors, are incapable of adapting to new situations, and scale poorly with problem size. As systems become more and more complex (with trends towards higher system densities and integration) new approaches need to be found. Sensor networks and robotic systems are two particular domains requiring innovative distributed approaches (e.g. [23, 33, 34]).

Not surprisingly then, in architecting solutions to address these issues, designers are looking to self-organised multi-agent systems. In general, this is because they display the key benefits of [23]:

- adaptability to change;
- robustness to failure and damage; and
- a high level of scalability.

Therefore, the ability to design self-organised systems is highly desirable in order to exploit these benefits. However as we describe in Sect. 2.5 most current approaches are ad-hoc (e.g. with genetic algorithms), either being specifically tailored to the given problem or not associated with a more widely used theoretical framework.

Indeed, this is part of the wider issue that complex systems science itself "lacks a common formal framework for analysis" [1]. The key goal of this science is to find general principles underlying classes of complex systems, and tools applicable across many systems from different domains. Certainly there have been successes here; arguably the greatest amongst these being the study of the topology or graph-theoretic properties of networks (see Sect. 2.4). With respect to time-series dynamics though, there is certainly a need for a widely accepted, rigorous and complete theoretical framework to be used. In the next section we describe information theory and its application to quantifying computation as a key candidate approach here.

2.2 Information Theory

Approaches considering computation and information are growing in popularity as candidate frameworks for the analysis and design of complex systems [1, 2, 35]. because commonalities in the handling of information Information theory is the language of computation and is at the centre of these approaches, which we describe in this section. We begin with an outline of the basic measures of information theory in Sect. 2.2.1. In Sect. 2.2.2 we then describe how these average measures can be extended to local dynamics in space and time, which is a key approach for this thesis. We also describe how to apply the measures to continuous variables in Sect. 2.2.3. We present support for the application of information theory to complex systems in Sect. 2.2.4, and describe several instances where information theory has already proven to be a useful framework for the design and analysis of complex systems. This leads us to highlight the opportunity for the establishment of a framework for quantifying the information dynamics of computation in complex systems.

2.2.1 Information-Theoretic Measures

Information theory was introduced by Shannon [36] to describe the transmission of information, incorporating concepts of reliability and efficiency. The basic concept of information theory is the communications channel, formed from an information source (which produces messages and encodes them onto a noisy and error prone channel) and a receiver (which decodes the channel's output in attempting to retrieve the original message). Information-theoretic quantities are formed by treating the specific messages x produced by the source as random variables X over a probability distribution function (PDF) $p(x)$ of the set of possible messages. Here we present the important information-theoretic quantities (see [37, 38] for further details). While

we introduce the quantities in discrete form in this section, we describe extensions to continuous variables in Sect. 2.2.3.

The fundamental quantity is the Shannon **entropy**, which represents the average uncertainty associated with any measurement x of a random variable X (units in bits):

$$H_X = -\sum_x p(x) \log_2 p(x). \tag{2.1}$$

Specifically, this quantifies the *average* number of bits needed to encode the random variable X, and can thus be seen as the information content of the particular message. This information content, or the "informative" nature of a message, can be interpreted as how surprising or unlikely its event was [39]. The Shannon entropy can also be interpreted as the level of diversity in the source [1].

The **joint entropy** of two (or more) random variables X and Y is a generalisation to quantify the uncertainty of the joint distribution of X and Y:

$$H_{X,Y} = -\sum_{x,y} p(x, y) \log_2 p(x, y). \tag{2.2}$$

The **conditional entropy** of X given Y is the average uncertainty that remains about x when y is known:

$$H_{X|Y} = -\sum_{x,y} p(x, y) \log_2 p(x \mid y), \tag{2.3}$$

$$= H_{X,Y} - H_Y. \tag{2.4}$$

Note that we have:

$$p(x \mid y) = p(x, y)/p(y). \tag{2.5}$$

The **mutual information** between X and Y measures the average reduction in uncertainty about x that results from learning the value of y, or vice versa:

$$I_{X;Y} = \sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p(x)p(y)}. \tag{2.6}$$

$$= H_X + H_Y - H_{X,Y}, \tag{2.7}$$

$$= H_X - H_{X|Y} = H_Y - H_{Y|X}. \tag{2.8}$$

From other perspectives, the mutual information can be said to measure how much information X and Y have in common, or how much information knowing the value of x tells one about the value of y on average. It can also be stated as the deviation of X and Y from independence (i.e. the Kullback–Leibler divergence of $p(x, y)$ from $p(x)p(y)$ [38]).

The mutual information can be generalised to a set of more than two variables as the **multi-information** or **integration** [40]. The multi-information is a

measure of the deviation from independence of the G components in the system $\mathbf{X} = \{X_1, X_2, \ldots, X_G\}$:

$$I_{\mathbf{X}} = I_{X_1;X_2;\ldots;X_G} = \left(\sum_{g=1}^{G} H_{X_g}\right) - H_{X_1,X_2,\ldots,X_G}. \tag{2.9}$$

The multi-information of a set $\mathbf{Z} = \{\mathbf{X}, \mathbf{Y}\}$ can be expressed iteratively in terms of the multi-information of its components individually and the mutual information between those components:

$$I_{\mathbf{Z}} = I_{\mathbf{X}} + I_{\mathbf{Y}} + I_{\mathbf{X};\mathbf{Y}}. \tag{2.10}$$

The **conditional mutual information** between X and Y given Z is the mutual information between X and Y when Z is known:

$$I_{X;Y|Z} = \sum_{x,y,z} p(x, y, z) \log_2 \frac{p(x \mid y, z)}{p(x \mid z)}. \tag{2.11}$$

$$I_{X;Y|Z} = H_{X|Z} - H_{X|Y,Z} = H_{Y|Z} - H_{Y|X,Z}. \tag{2.12}$$

It can also be stated as the average common information between X and Y that was not contained in Z. This is the only valid "three-term entropy" [38] (though any of X, Y or Z can be joint variables). It is important to note that the three term expression can defy our intuition. In particular, though we might naively expect the conditional mutual information $I_{X;Y|Z}$ to always be smaller than the mutual information $I_{X;Y}$, it is also possible for $I_{X;Y|Z}$ to be larger. As described in [38], an example is where X, Y and Z are the input, noise and output from a binary symmetric channel. If the noise and input are independent $I_{X;Y} = 0$, but we will find that $I_{X;Y|Z} > 0$ because knowing the output provides us with some information about the previously unknown relationship between the input and noise.

The **channel capacity** is the maximum amount of information that a received signal Y can contain about a signal X transmitted through the channel. Since the information that the received signal contains about the transmitted signal is represented by their mutual information, channel capacity is specifically defined as the maximum mutual information for the channel over all distributions of the transmitted signal:

$$C(p(y \mid x)) = \max_{p(x)} I_{X;Y}. \tag{2.13}$$

This renders channel capacity as asymmetric and causal (in contrast to mutual information) [41]. Also in contrast to mutual information, it is a property of the channel itself rather than a property of the dynamics for a specific interaction over the channel.

The **entropy rate** is the limiting value of the rate of change of the joint entropy over k consecutive states of X, (i.e. measurements $x^{(k)}$ of the random variable $X^{(k)}$),

as k increases [39]:

$$H_{\mu X} = \lim_{k\to\infty} \frac{H_{X^{(k)}}}{k} = \lim_{k\to\infty} H'_{\mu X}(k), \tag{2.14}$$

$$H'_{\mu X}(k) = \frac{H_{X^{(k)}}}{k}. \tag{2.15}$$

The entropy rate can also be expressed as the limiting value of the conditional entropy of the next state of X (i.e. measurements x_{n+1} of the random variable X') given knowledge of the previous k states of X (i.e. measurements $x_n^{(k)} = \{x_{n-k+1}, \ldots, x_{n-1}, x_n\}$, up to and including time step n, of the random variable $X^{(k)}$):

$$H_{\mu X} = \lim_{k\to\infty} H_{X'|X^{(k)}} = \lim_{k\to\infty} H_{\mu X}(k), \tag{2.16}$$

$$H_{\mu X}(k) = H_{X^{(k+1)}} - H_{X^{(k)}}. \tag{2.17}$$

We note that while these limiting values exist as $k \to \infty$ for stationary processes (e.g. see [37]), there is no guarantee that such limits exist for non-stationary processes.

Grassberger [42] first noticed that a slow approach of the entropy rate to its limiting value was a sign of complexity. Formally, Crutchfield and Feldman [39] use the conditional entropy form of the entropy rate (2.16)[4] to observe that at a finite block size k, the difference $H_{\mu X}(k) - H_{\mu X}$ represents the information carrying capacity in size k-blocks that is due to correlations. The sum over all k gives the total amount of structure in the system, quantified as **excess entropy**[5] (measured in bits):

$$E_X = \sum_{k=0}^{\infty} \left[H_{\mu X}(k) - H_{\mu X} \right]. \tag{2.18}$$

The excess entropy can also be formulated as the mutual information between the semi-infinite past and semi-infinite future of the system:

$$E_X = \lim_{k\to\infty} E_X(k), \tag{2.19}$$

$$E_X(k) = I_{X^{(k)};X^{(k^+)}} \tag{2.20}$$

where $X^{(k^+)}$ is the random variable (with measurements $x_{n+1}^{(k^+)} = \{x_{n+1}, x_{n+2}, \ldots, x_{n+k}\}$) referring to the k future states of X (from time step $n+1$ onwards). This interpretation is known as the **predictive information** [43], as it highlights that the excess entropy captures the information in a system's past which can be used to

[4] $H_{\mu X}(k)$ here is equivalent to $h_\mu(L-1)$ in [39]. This means the sum in Eq. (2.18) starts from $k = 0$ as equivalent to $L = 1$.

[5] The excess entropy was labelled the "effective measure complexity" by Grassberger in [42].

predict its future. This is significant as it is explicitly consistent with the interpretation of the excess entropy as the amount of structure or memory in the system.

2.2.2 Localising Information-Theoretical Measures

Information-theoretic variables are generally defined and used as an *average* uncertainty or information. We are interested in considering *local* information-theoretic values, i.e. the uncertainty or information associated with a *particular observation* of the variables rather than the average over all observations.[6] Local measures within a global average are known to provide important insights into the *dynamics* of non-linear systems [45]. Indeed, the ability to investigate *time-series dynamics* of complex systems provides an important connection from information theory to *dynamical systems theory* or *non-linear time-series analysis* (e.g. see [46, 47]). Importantly, only local information-theoretic measures can describe the *dynamics of computation*, since they alone can describe how information is being manipulated at each step in time. In this section we define how to obtain these local values and describe their meaning.

We use the mutual information as an illustrative example, though note that the derivation applies equally to the other terms defined in Sect. 2.2.1. The (average) mutual information defines the average information in X about Y or vice-versa; localisations consider how much information is conveyed by specific observations or realisations x_n and y_n of the variables X and Y at time step n.[7]

First, we note that the mutual information $I_{X;Y}$ is defined in Eq. (2.6) as a sum over all possible state tuple observations $\{x_n, y_n\}$, weighted by the probability $p(x_n, y_n)$ of observing each such tuple. This probability $p(x_n, y_n)$ is operationally equivalent to the ratio of the count of observations $c(x_n, y_n)$ of $\{x_n, y_n\}$, to the total number of observations N made: $p(x_n, y_n) = c(x_n, y_n)/N$. To precisely compute this probability, the ratio should be composed over all realisations of processes of the observed variables (as described in [50]); realistically however, estimates will be made from a finite number of observations. Subsequently, we replace the count by its definition

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[6] *Local* information-theoretic measures are known as *point-wise* measures elsewhere [44].

[7] Appendix A describes two different approaches that have been presented to quantifying *partial* localisations of the mutual information $I(x_n; Y)$ [48]. The partial localisation $I(x_n; Y)$ considers how much information $I(x_n; Y)$ a specific value x_n at time step n gives about what value Y *might* take. We note this is distinct from the *full* localisations $i(x_n; y_n)$ that we consider here; this quantifies the amount of information conveyed by a specific value x_n about the specific value y_n that Y *actually takes* at time step n (or vice-versa). Our interest lies in these full localisations $i(x_n; y_n)$, as they quantify the specific amount of information involved or manipulated in the dynamics of the computation at time step n with the given realisation $\{x_n, y_n\}$. Appendix A demonstrates that there is only one approach to quantifying full localisations $i(x_n; y_n)$ that fulfils both additivity and symmetry properties.

We also note that a similar approach to "localising" information-theoretic values is by using sliding windows of observations (e.g. [49]). While this does provide a more local measure than averaging over all available observations, it is not local in the same sense as the term is used here (i.e. it does not look at the information involved in the computation at a *single specific time step*).

$c(x_n, y_n) = \sum_{g=1}^{c(x_n,y_n)} 1$, leaving the substitution $p(x_n, y_n) = \left(\sum_{g=1}^{c(x_n,y_n)} 1\right)/N$ into Eq. (2.6):

$$I_{X;Y} = \frac{1}{N} \sum_{x_n, y_n} \left(\sum_{g=1}^{c(x_n,y_n)} 1 \right) \log_2 \frac{p(x_n, y_n)}{p(x_n)p(y_n)}. \tag{2.21}$$

The log term may then be brought inside this inner sum:

$$I_{X;Y} = \frac{1}{N} \sum_{x_n, y_n} \sum_{g=1}^{c(x_n,y_n)} \log_2 \frac{p(x_n, y_n)}{p(x_n)p(y_n)}. \tag{2.22}$$

This leaves a double sum running over each actual observation g for each possible tuple observation $\{x_n, y_n\}$. This is equivalent to a single sum over all N observations:

$$I_{X;Y} = \frac{1}{N} \sum_{n=1}^{N} \log_2 \frac{p(x_n, y_n)}{p(x_n)p(y_n)}. \tag{2.23}$$

It is clear then that the mutual information measure is an *average* (or expectation value) of a *local mutual information* at each observation:

$$I_{X;Y} = \langle i(x_n; y_n) \rangle_n ; \tag{2.24}$$

$$i(x_n; y_n) = \log_2 \frac{p(x_n, y_n)}{p(x_n)p(y_n)}. \tag{2.25}$$

Note that by convention we use lower-case symbols to denote local values throughout this thesis.

The measure is *local* in that it is defined at each time step n. This method of forming a local information-theoretic measure by extracting the log term from a globally averaged measure is applicable to any of the aforementioned information-theoretic variables. For example, the conditional mutual information $I_{X;Y|Z}$ in Eq. (2.11) can also be expressed as the average of a *local conditional mutual information* $i(x_n; y_n \mid z_n)$ at each observation n:

$$I_{X;Y|Z} = \langle i(x_n; y_n \mid z_n) \rangle_n ; \tag{2.26}$$

$$i(x_n; y_n \mid z_n) = \log_2 \frac{p(x_n \mid y_n, z_n)}{p(x_n \mid z_n)}. \tag{2.27}$$

We also note that local mutual information values (including conditional ones) can be negative. This occurs where in Eq. (2.25) for example, $p(x_n, y_n) < p(x_n)p(y_n)$; i.e. there is more uncertainty $p(x_n \mid y_n)$ in x_n given y_n than there was uncertainty $p(x_n)$ in x_n independently of knowing y_n. These negative values are actually quite meaningful, and can be interpreted as there being negative information in the value of y_n about x_n. We could also interpret the value y_n as being *misleading* or

misinformative about the value of x_n, because it had *lowered* our expectation of observing x_n prior to that observation being made in this instance. Importantly, it is not possible for local entropies $h(x_n) = -\log_2 p(x_n)$ to become negative.

The technique has been used (less explicitly) for the local excess entropy [50], the local statistical complexity [50, 51], and the local information [52]. Despite some interest from these authors [50–52], relatively little exploration has been made into the dynamics of these local information measures in complex systems, and certainly none has been made into the local dynamics of information storage, transfer and modification.

2.2.3 Information-Theoretic Measures of Continuous Variables

The information-theoretic measures in Sect. 2.2.1 are defined for discrete variables X and Y. In the discrete domain, measuring the relevant PDFs $p(x)$, $p(y)$ and $p(x, y)$ from a set of observations $\{x, y\} \in \{X, Y\}$ is straightforward. This is not the case for continuous variables X and Y. While it is possible to define the information-theoretic measures in integral form (see Chap. 9 of [37]), this requires the PDFs to be well-defined on the range of X and Y which is not the case for a finite number of samples.

A simple approach to handling continuous variables is to discretise the observations. One can then apply the relevant standard information-theoretic calculation to the discretised values. However with a slight increase in effort, one can remain in the continuous regime and so maximise the incorporation of the subtle features of the data into the calculations.

To do so in computing the transfer entropy[8] Schreiber [53] recommends using *kernel estimation* to estimate the required probabilities. This recommendation was used for example to compute transfer entropy in signal transduction by calcium ions in [54]. Kernel estimation relies on the notion that the given measure is *necessarily* an average of the local values of the measure in time (as per Sect. 2.2.2) rather than over all possible state transition tuples [47, 53]. First, the approach assumes that the required PDFs can be defined for each observation $n \in [1, N]$ in our set. For example, for the mutual information $I_{X;Y}$ the approach assumes that the PDFs for each observation ($\hat{p}_r(x_n, y_n)$, $\hat{p}_r(x_n)$ and $\hat{p}_r(y_n)$) can be defined, with the measure computed as an average over local values:

$$I_{X;Y} = \frac{1}{N} \sum_{n=1}^{N} \log_2 \frac{\hat{p}_r(x_n, y_n)}{\hat{p}_r(x_n)\hat{p}_r(y_n)}. \tag{2.28}$$

[8] The transfer entropy is arguably the most important measure used in this thesis. As such, recommendations on approaches to compute it will be followed for other measures also for consistency. The transfer entropy will be introduced in Chap. 4

The method then computes the probabilities for each observation $n \in [1, N]$ by counting the number of "similar" observations; the similarities are calculated using a kernel function Θ to judge "similarity" and a resolution r. For example, the probability of an observation (x_n, y_n) is estimated as:

$$\hat{p}_r(x_n, y_n) = \frac{1}{N} \sum_{n'=1}^{N} \Theta\left(\left| \begin{pmatrix} x_n - x_{n'} \\ y_n - y_{n'} \end{pmatrix} \right| - r \right). \tag{2.29}$$

By default Θ is the step kernel ($\Theta(x > 0) = 0$, $\Theta(x \leq 0) = 1$), and the norm $|\cdot|$ is the maximum distance. This combination results in $\hat{p}_r(x_n, y_n)$ being the proportion of the N values which fall within r of $\{x_n, y_n\}$ in both dimensions X and Y. Other choices for the kernel Θ and the norm $|\cdot|$ are possible. Conditional probabilities may be defined in terms of their component probabilities following Eq. (2.5). Importantly, a mutual information computed by kernel estimation is a slightly different quantity to one measured on discrete values: here the quantity measures the average reduction in uncertainty about predicting x *within* r that results from learning the value of y *within* r.

We note the existence of various techniques that attempt to improve upon kernel estimation, e.g. introducing optimisations in time or attempting to reduce errors. The most relevant of these are the techniques of Kraskov et al. [55, 56] for estimating the mutual information. This approach uses two specific enhancements designed to reduce errors when handling a small number of observations. The first is the use of Kozachenko-Leonenko estimates [57] of log-probabilities via nearest-neighbour counting; the second is to use a fixed number K of nearest-neighbours in the joint probability space (see [55, 56] for further details). This can be thought of as a kernel-estimation type technique, with a dynamically altered kernel width to adjust to the density of samples in the vicinity of any given observation. These adjustments smooth out errors in the PDF estimation. They also alter the meaning of the measure to being the information one learns from y about predicting the value of x within the nearest K values of $\{x, y\}$.

2.2.4 Reasons for Application to Complex Systems

Information theory is gaining popularity as an analytic tool for complex systems [1, 2, 58]. At first glance, obvious areas of its applicability lie in characterising order and disorder, since these have clear parallels to the concept of entropy [2]. The reasons for its application go beyond this however.

This highly *abstract nature* of information theory means that it makes no assumptions about the system. The only requirement for the application of information theory is *probability distributions* [21], which are generally accessible as they are a natural way of describing the dynamics of complex systems. This also means information-theoretic tools can handle *stochastic* as well as deterministic systems

[21]. The *generality* offered by information theory is crucial in complex systems analysis, where systems from all realms and varieties of science are candidates for investigation. Such generality means that analytic tools developed for one system are applicable to other superficially unrelated systems; this is a key goal of complex systems science.

Furthermore, information theory provides definitions which can be formulated mathematically, and its theoretical basis is well-developed, sound and stable [1]. Also, it captures *non-linear* relationships which we know to be a vital feature for any tool in complex systems science. Prokopenko et al. [1] claim that application of information theory to complex systems simply involves identification of the appropriate information channels, followed by computation of the relevant measures.

We must however note some arguments against the application of information theory here. Gibson [59] claimed that information theory was not applicable to natural systems as the environment does not send messages to living systems or agents. It is generally accepted now however that information theory does not need *explicit* messages on which to operate; its applicability comes in examining probability distributions of agent states and *implicit* messaging or influences. This is akin to the notion of *intrinsic computation* [19, 60]. Another criticism is that information-theoretic measures typically require a large amount of data in order to be accurately calculated. This is true to an extent, however modern computing tools (e.g. PCs) can handle typical data sets and generate meaningful results with ease, and techniques such as those of Kraskov et al. [55, 56] discussed in Sect. 2.2.3 have reduced the data size requirements.

From our perspective, information is the language of computation, so any attempt to characterise the dynamics of distributed computation, or to describe complex systems in these terms, must necessarily involve information theory. This is best captured by Gell-Mann [35]: "Although (complex adaptive systems) differ widely in their physical attributes, they resemble one another in the way they handle information. That common feature is perhaps the best starting point for exploring how they operate." This statement alludes to three important features of the information-theoretical perspective: its generality, focus on computation, and the known useful insights it produces. We will discuss the known useful insights it produces in the next sub-sections.

2.2.4.1 Examples of Information-Theoretic Analysis of Complex Systems

Information theory has already proven to be a useful framework for the design and analysis of complex self-organised systems [1]. As suggested above, it has been used to *characterise order and disorder*. Obviously the Shannon entropy can be used for this purpose, but this can be misleading (e.g. returning high values for periodic processes). More sophisticated is the use of the entropy rate in order to measure uncertainty in the context of the system's past, e.g. [19, 39].

Information theory has also been applied to the notoriously difficult task of attempting to *measure complexity* [1]. Various perspectives used here include the

amount of information required to describe or predict the system or time required to implement it, or richness of the dynamics. General intuition is that completely statically ordered processes (with fixed states) and completely disordered or random process (with completely unpredictable states) have least complexity [11] and most perspectives aim to capture this. One common theme is the uncertainty (entropy) of system specific measures (e.g. perturbation avalanches in boolean networks [61]), or more generally the variance in information-theoretic measures (e.g. the variance of the *input entropy* with time [62]). Also, the Tononi-Sporns-Edelman (TSE) complexity [40] measures the extent to which a set of variables exhibit both global integration and at the same time functional segregation. It can be analytically related to spatial formulations of the excess entropy [63, 64].

Perhaps most prominent though is the *statistical complexity* [20, 65, 66]. Found in the study of *computational mechanics*, it attempts to capture the computational effort required to model complex behaviour. Formally, it is the uncertainty in the internal state of a minimal state machine (known as an ϵ-machine) that can statistically mimic a given process. This internal state is known as a *causal state*; two (possibly observationally different) states of a system have an equivalent causal state if the PDF of their future states is statistically equivalent. The ϵ-machine can be constructed for single agents, for system-wide collections of agents, or with a hybrid approach known as the *light-cone formulation*. The light-cone formulation forms causal states between past and future light cones centred at a given time step for a given agent. A *past light-cone* is the set of states of that agent and others that are causal ancestors of the given space-time point. A *future light-cone* contains the corresponding causal descendants.

A popular application of information theory has been the *investigation of order-chaos phase transitions* and whether measures of complexity are maximised in these transitions. These are effectively quantitative studies of the *edge of chaos* hypothesis [15, 16], which suggests that complexity is maximised in order-chaos phase transitions and in particular that computational properties are maximised there. For example, information-theoretic measures of complexity are demonstrated to be maximised at a critical phase in the construction of self-organised impact boundaries in [32], reflecting phase transitions established using task-specific measures. Other examples that suggest maximisations of information-theoretic measures of complexity in such phase transitions include [17, 67]. Some of the most relevant work regarding of the edge of chaos hypothesis considers cellular automata (these are discussed in Sect. 2.3.3). Importantly, other information-theoretic studies [19, 68] have demonstrated that there is no *universal* complexity trend through such phase transitions (in particular for more complicated phase transitions), and that computation can and does take place elsewhere on the order-chaos continuum.

With measures for complexity in place, *measuring self-organisation* as a change in complexity follows its definition in Sect. 2.1.2 and is typical, e.g. [20–22]. This is done for example by Shalizi et al. [20] using the statistical complexity. Similarly, complexity measures are also useful in quantifying *emergence*, e.g. see [50, 66].

Information theory has also been applied to the analysis of topological structure. Although such structure is static and contains no time-series dynamics, the measures

are made on "observations" of the structure at each node or link in the network. For example, the amount of information the degree of nodes on either end of a given link have in common is considered in [69, 70].

Also, information-theoretic measures are being increasingly used to guide the design of artificial self-organised systems. This approach is discussed in detail in Sect. 2.5.

2.2.4.2 Information Dynamics of Computation in Complex Systems

It is important to observe that the applications described above go well beyond measuring only order and disorder, but address the nature of computation in complex systems. Many focus on the overall complexity of such computation. Our interest is in the information dynamics supporting such complex computation: how information is *stored*, *transferred* and *modified* in the system. Such concepts are considered most prominently in cellular automata, the most important venue for discussions of computation in complex systems. As we will discuss in detail in Sect. 2.3, a clear *qualitative* understanding of the dynamics of information in cellular automata has been established, but never validated with *quantitative* measures.

The lack of a definitive framework for these operations on information has impacts beyond cellular automata and across complex systems science. Their importance is underlined in that each of these operations has been considered in various complex systems settings. Information storage is considered a crucial part of the dynamics of human brain networks [71], synchronisation between coupled systems [72], coordinated motion in modular robots [73], and in the dynamics of inter-event distribution times [74]. Information transfer is manifested in "information cascades" spreading across schools of fish [75], said to give rise to self-organisation via dipole-dipole interactions [76], and optimum efficiency of information transmission is said to underpin order-chaos phase transitions in ant foraging [17]. Information modification is a key operation in collision-based computing models (e.g. [77, 78], including soliton dynamics and collisions [79]), as well as in biological neural networks and models thereof [80–83]

However, the lack of clearly established measures for these information dynamics has led to unanswered speculation on their role in complex computation. This is particularly true regarding information transfer, e.g. the conflicting suggestions that transfer is maximised in complex dynamics at a phase transition between ordered and chaotic behaviour [17, 67], or alternatively is at an intermediate level with maximisation leading to chaos [15, 84]. This lack of clarity has also led to the concepts of information transfer and causal effect sometimes being unfortunately directly equated, e.g. [85–87]. It also extends to the study of dynamics on networks, where as we will discuss in Sect. 2.4 understanding information transfer "is one of the most important open problems in science" [2].

Certainly there are good candidate measures for some of these concepts, notably the excess entropy [39] for measuring information storage (see Eq. (2.18)) and

transfer entropy for information transfer (to be discussed in Chap. 4). However, their local dynamics have not been investigated, nor have they been demonstrated to align with accepted instances of these concepts in the dynamics of computation (see computation in CAs in Sect. 2.3.3). While brief explorations of local information-theoretic values values have been made (see Sect. 2.2.2), there is no established approach to studying the information dynamics of computation that aligns with dynamical systems theory. In particular, we have no understanding of how the component operations of information dynamics relate to each other. What we seek here are not simply more complexity measures, but a methodology to explain the dynamics of how complex distributed computation occurs.

In the following sections, we discuss in more detail the need for a framework for information dynamics in a number of specific application domains: cellular automata, the dynamics of networks, and guided self-organisation.

2.3 Cellular Automata

Cellular automata (CAs) are an important general class of models, since they support complex computation and provide the ability to model complex systems in nature [88]. They are *the most important domain for the study of distributed computation*, as the subject of a large amount of related work regarding the nature of computation in complex systems (e.g. [15, 51, 52, 62, 88–95]). Significantly, Von Neumann was known to be a strong believer that "a general theory of computation in 'complex networks of automata' such as cellular automata would be essential both for understanding complex systems in nature and for designing artificial complex systems" ([88] describing [96]). We select CAs for experimentation here for these reasons and because there is very clear *qualitative* observation of emergent structures representing information storage, transfer and modification therein (e.g. [15, 88]).

In this section, we describe the mechanics of CAs in Sect. 2.3.1, then discuss interpretations of complex behaviour and emergent structures in CAs in Sect. 2.3.2. We explore the perspective of computation within CAs and outline opportunities for providing quantitative insights on the nature of such computation in Sect. 2.3.3. The reader is introduced to several of the important CA rules investigated in this thesis in Sect. 2.3.4. We also describe existing techniques for filtering emergent structure in CAs in Sect. 2.3.5, outlining how the ability to filter such structure from a computational perspective would provide a novel contribution.

2.3.1 Functionality of Cellular Automata

CAs are discrete dynamical lattice systems. They consist of an array of cells which each update their discrete state as a function of the states of a fixed number of spatially neighbouring cells using a uniform rule. These updates occur synchronously

in discrete time. Although the behaviour of each individual cell is very simple, the (non-linear) interactions between all cells can lead to very intricate global behaviour. As such, CAs have become a classic example of self-organised complex behaviour. Of particular importance, CAs have been used to model real-world spatial dynamical processes, including fluid flow, earthquakes and biological pattern formation [88].

The neighbourhood used as inputs to a cell's update rule at each time step is usually some regular configuration. In 1D CAs, this means the same range r of cells on each side and includes the current state of the updating cell. One of the simplest variety of CAs—1D CAs using binary states, deterministic rules and $r = 1$ neighbour on either side—are known as the *Elementary CAs*, or *ECAs*. As such, the parent nodes which determine $x_{i,n+1}$ (the value of node X_i at time $n + 1$) in an ECA are $\{x_{i-1,n}, x_{i,n}, x_{i+1,n}\}$. The past light-cone of $x_{i,n+1}$ is then made up of these nodes, their parents, and so on. The future light-cone of $x_{i,n+1}$ consists of the nodes it has a direct causal effect on, i.e. $\{x_{i-1,n+2}, x_{i,n+2}, x_{i+1,n+2}\}$, the nodes they have a direct causal effect on, and so on.

Example evolutions of ECAs from random initial conditions may be seen in Fig. 2.1 for the examples we discuss in Sect. 2.3.4. For more complete definitions of CAs, including the definition of the Wolfram rule number convention for specifying update rules, see Ref. [92].

2.3.2 Complex Behaviour in Cellular Automata

Wolfram [89, 92] sought to classify CA rules in terms of their asymptotic behaviour in time. He proposed four classes of asymptotic behaviour: I. Homogeneous state; II. Simple stable or periodic structures; III. Chaotic aperiodic behaviour; IV. Complicated localised structures, some propagating. This classification has been highly influential on subsequent research due to the parallels (for discrete state and time systems) with our knowledge of dynamical systems. Here classes I and II represent *ordered* behaviour, and class III represents *chaotic* behaviour. Class IV represents *complex* behaviour and is considered to lie between the ordered and chaotic classes.

The analogy to dynamical systems is taken further in considering the *state-space* of the global state $\mathbf{X}$ of the CA, and the *attractors*[9] and *transient* paths[10] in this space [62]. This discrete state-space is analogous to Poincaré's *phase portrait* in continuous dynamics. Ordered CAs are said to exhibit short transient paths with high convergence to attractors. Chaotic CAs exhibit long transient paths with low

[9] An attractor is a single global state $\mathbf{x}_n = \{\ldots, x_{i-1,n}, x_{i,n}, x_{i+1,n}, \ldots\}$ or periodic sequence of global states that a CA (generally considering fixed sizes) can reach after a finite number of time steps but then never leave (unless some stochasticity is introduced into the dynamics).

[10] A transient is a path of global states that a CA could traverse before reaching an attractor. For deterministic CAs, two or more transient paths can converge on the same next state, but a given state cannot diverge via multiple transient paths to more than one next state. A CA of finite size C cells must reach an attractor after a finite number of time steps (since there are a finite number b^C of possible global states, where b is the base or number of possible discrete states for each cell).

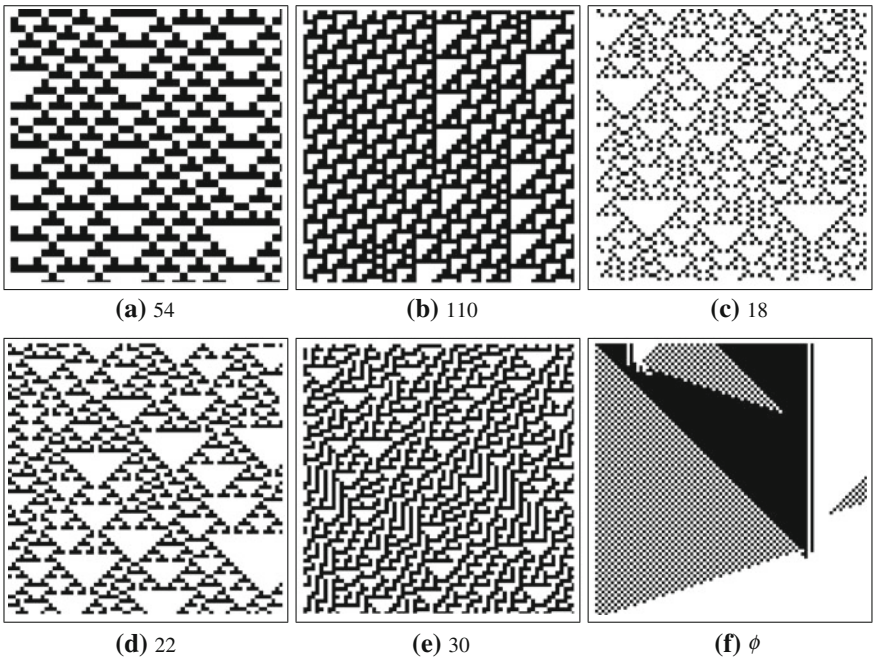
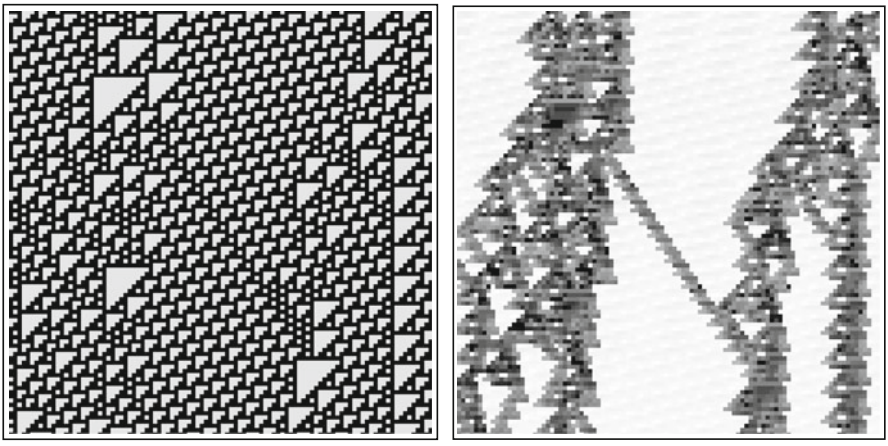


Fig. 2.1 Example time-evolutions of states of several important CA rules from random initial conditions. All rule numbers follow the Wolfram rule number convention [92], except rule ϕ which is an $r = 3$ CA evolved to classify the density of initial states (see Sect. 2.3.4). Black cells are in the "1" state, white cells are in the "0" state. Time increases down the page for all CA plots: one row represents one time slice $\mathbf{x}_n$ of the states of each cell in the CA. Note: the first row shown here is typically for some time $n > 0$ (NB: (a)–(c) Reprinted with permission from J. T. Lizier et al. [97]. Copyright 2010, American Institute of Physics. (d)–(e) are reprinted from Ref. [98] with permission of Springer.)

convergence to attractors. In contrast, complex CAs are observed to exhibit maximal uncertainty in the length of transient paths.

Much conjecture remains as to whether Wolfram's classes are quantitatively distinguishable however (e.g. see [99]). Regardless of this however they certainly provide an interesting qualitative analogy to dynamical systems.

More importantly, the approach seeks to characterise complex behaviour in terms of *emergent structure* in CAs: *particles*, *gliders*, *blinkers*, *domains* and *domain walls*. Qualitatively, a domain may described as a set of background configurations in a CA, for which any given spatially-extended configuration will update to another such configuration in the set in the absence of any disturbance. Domains are formally defined within computational mechanics [94] as spatial process languages in the CA. Particles are qualitatively considered to be elements of *coherent* spatiotemporal structure which disturb background domains. Gliders are particles which repeat periodically in time while moving spatially, while repetitive non-moving particle



(a) **(b)**

Fig. 2.2 Example of filtering of CA rule 110: **a** raw states; **b** local statistical complexity [51] of these states, which clearly highlights gliders and blinkers against the background domain. These diagrams were produced using the CimulA package [100], then rotated to have cells across and time down the page (matching Fig. 2.1)

structures are known as *blinkers*.[11] Formally, particles are defined within computational mechanics as a boundary between two domain regions [94]; as such, they can also be termed as domain walls, though this is typically used with reference to aperiodic particles.

These emergent structures are more clearly visible when the CA is *filtered* in some way, e.g. see local statistical complexity [51] applied to ECA rule 110 in Fig. 2.2. We discuss techniques for such filtering in Sect. 2.3.4. First, we need to understand how distributed computation is studied and measured in CAs.

2.3.3 Computation in Cellular Automata

CAs can be interpreted as undertaking distributed computation: it is established that "data represented by initial configurations is processed by time evolution" [89]. Indeed, any time evolution of the CA represents an intrinsic computation [19, 60] in determining its own future and ultimate attractor state and phase, in the same way that the universe "computes itself" [101].

As such, computation in CAs has been a popular topic for study (see [88]), with a particular focus in observing or constructing (Turing) universal computation in certain CAs. An ability for universal computation is defined to be where "suitable initial

[11] Of course, blinkers can be considered to be vertical or non-moving gliders, while both are particles.

configurations can specify arbitrary algorithm procedures" in the distributed computing entity, which is capable of "evaluating any (computable) function" [89]. Wolfram conjectured that all class IV complex CAs were capable of universal computation [89, 90]. He went on to state that prediction in systems exhibiting universal computation is limited to explicit simulation of the system, as opposed to the availability of any simple formula or "short-cut" [89, 90]. He drew parallels to undecidability in the halting problem for universal Turing machines which are echoed by Langton [15] and Casti [102]. Casti extended the analogy to undecidable statements in formal systems (i.e. Gödel's Theorem). The undecidability is also linked back to the maximal uncertainty in transient lengths for complex or critical behaviour in other systems (e.g. see [103]). The capability for universal computation has been proven for several CA rules, through the design of rules generating elements to (or by identifying elements which) specifically provide the component operations required for universal computation: **information storage**, **transmission** and **modification**. Examples here include most notably the 2D rule known as the "Game of Life" [104] and ECA rule 110 [105]; also see [106] and discussions in [88].

The focus on elements providing information storage, transmission and modification pervades discussion of all types of computation in CAs (e.g. see also "collision-based computing" in [78, 107]). Wolfram claimed that in class III CAs information propagates over an infinite distance at a finite speed, while in class IV CAs information propagates irregularly over an infinite range [90]. Langton [15] hypothesised that complex behaviour in CAs at the *edge of chaos* exhibited the three component operations required for universal computation. He suggested that the more chaotic a system becomes the more information transmission increases, and the more ordered a system becomes the more information it stores. Complex behaviour was said to occur at a phase transition between these extremes requiring an *intermediate level* of both information storage and transmission. He suggested for example, that intermediate levels of information transmission can support large correlation lengths, but if information propagates too well, *coherent information decays into noise*.

Importantly, Langton elaborates [15] that transmission of information means that the "dynamics must provide for the propagation of information in the form of signals over arbitrarily long distances". He goes on to suggest that **particles form the basis of information transmission**, since they appear to facilitate communication about the dynamics in one area of the CA to another area. To complete the qualitative identification of the elements of computation in CAs, he also suggested that **blinkers formed the basis of information storage**, and **collisions** between propagating (particles) and static structures (blinkers) "**can modify either stored or transmitted information in the support of an overall computation**" or decision process about the dynamics. He also made rudimentary attempts at quantifying the average information transfer (and to some extent information storage), via mutual information.[12] Recognising *the importance of the emergent structures to computation*, several examples exist of attempts to automatically identify CA rules which give rise to particles

[12] However as discussed in Chap. 4 this is a symmetric measure not capturing directional transfer.

and gliders, e.g. [62, 108], suggesting these to be the most interesting and complex CA rules.

Several authors however criticise the aforementioned approaches of attempting to classify CAs in terms of their generic behaviour or "bulk statistical properties", suggesting that the wide range of differing dynamics taking place across the CA makes this problematic [88, 94]. Related here is that, despite Langton's hypothesis and the introduction of useful measures of the complexity of CA rules (e.g. [19, 51, 62]), there is no established phase transition dictated by a single order-chaos parameter in CAs [19]. Also, Gray suggests that there there may indeed be classes of CAs capable of more complex computation than universal computation alone [99]. More importantly, Hanson and Crutchfield [94] criticise the focus on universal computational ability as drawing away from the ability to identify "generic computational properties", i.e. a lack of ability for universal computation does not mean a CA is not undertaking any computation at all.

Alternatively, these studies suggest that analysing the rich space-time dynamics *within* the CA is a more appropriate focus. As such, references [88, 94] and others have analysed the *local* dynamics of intrinsic or other specific computation, focusing on the computational roles of emergent structures. They align with Langton's observations of blinkers storing information, particles facilitating the transfer of information and collisions facilitating the information modification or processing. Notable examples here include: the method of applying filters from the domain of computational mechanics by Hanson and Crutchfield [94]; and analysis using such filters to analyse CA rules selected via evolutionary computation to perform classification tasks by Mitchell et al. [93, 109]. Also relevant are studies which deeply investigate the nature of particles and their interactions, e.g.: particle types and their interaction products identified for particular CAs in [109–112], rules established for their interaction products in [113], and studies of "collision-based computing" in [78, 107].

This perspective of the computational roles of emergent structures is important not only for our theoretical understanding of the nature of distributed computation, but is also important on a practical level. This is because it has been used to explain the computational function of similar propagating coherent emergent structures and their interactions in neural circuits [3] and in the opening and closing of stomatal apertures in plants [4].

Despite such interest, there is no quantitative evidence to support these conjectures about the role of emergent structures in computation in CAs. Simply quantifying the average information dynamics in CAs would be novel itself; we are not aware of any previous direct measurement of information storage, transfer or modification in CAs.[13] However measuring *averages* will not suffice: e.g. a coincidence of

[13] Langton's use of mutual information in [15] is a symmetric measure not capturing directional transfer. As we will describe in Sect. 3.1, Grassberger inferred the excess entropy to be infinite in some CAs under certain circumstances by studying trends of the entropy rate [42, 114]. The study did not make any direct measurements of the excess entropy apart from these inferences, and focussed on collective excess entropy. Crutchfield and Feldman measure the excess entropy

high information transfer in CAs with gliders would not mean that gliders are the information transfer agents. Providing such evidence requires the ability to quantify the information dynamics of computation on a *local scale in space and time* within the CA. It is only the local scale that will explicitly show the computational roles of the emergent structures. Yet there is no complete framework that locally quantifies the individual information dynamics of distributed computation within CAs or other systems. In this thesis we will provide such a framework in Chaps. 3–5, and use to it to describe how the component operations of computation interact to give rise to emergent complex behaviour. We expect this framework to highlight blinkers and domain regions as dominant information storage processes, particles (including gliders and domain walls) as information transfer, and particle collisions as information modification. This will make a significant contribution to our fundamental understanding of the nature of distributed computation.

2.3.4 Examples of Distributed Computation in CAs

In this thesis, we will examine the computation carried out by several important ECA rules:

- Class IV complex rules 110 and 54 [92], both of which exhibit a number of glider types and collisions. See raw states in Fig. 2.1a, b, and a filtered view of the gliders in rule 110 in Fig. 2.2. ECA rule 110 is the only proven computationally universal ECA rule [105].
- Rules 22 and 30 as representative class III chaotic rules [92] (see Fig. 2.1d, e).
- Rules 18 as a class III rule which contains domain walls against a chaotic background domain [91, 94] (see Fig. 2.1c).

While these ECAs are not computing any particular human understandable task, the intrinsic computation they are undertaking is of interest.

We also examine a CA carrying out a "human-understandable" computational task. ϕ_{par} is a 1D CA with range $r = 3$ (Wolfram rule number 0xfeedffdec1aaeec0-eef000a0e1a020a0). It resulted from an evolutionary computation experiment by Mitchell et al. [93, 109] which aimed to classify whether an initial CA configuration had a majority of 1's or 0's by reaching a fixed-point attractor of all 1's for the former or all 0's for the latter. This CA rule achieved a success rate above 70 % in its task. An example run of this CA can be seen in Fig. 2.1f. The CA appears to have evolved to carry out this computation using blinkers and domains for information storage, gliders for information transfer and glider collisions for information modification. The computation has been interpreted as follows [93, 109]. The CA exhibits an initial emergence of domain regions of all 1's or all 0's storing information about local high densities of either state. Where these domains meet, a checkerboard domain

(Footnote 13 continued)
in *spatially*-extended blocks in various CAs in an ensemble study in [19], however this is not an information storage since the measurement is not made *temporally*.

propagates slowly (1 cell per time step) in both directions, with the gliders at the leading edge transferring information regarding a *soft* uncertainty about the density in this part of the CA. Some "certainty" is provided where the checkerboard encounters a blinker boundary between 0 and 1 domains, which stores information about a *hard* uncertainty in that region of the CA. This results in an information modification event where the domain on the opposite side of the blinker to the incoming checkerboard is concluded to represent the higher density state, and is allowed to propagate over the checkerboard domain. Because of the greater certainty attached to this decision, this new information transfer occurs at a faster speed (3 cells per time step); it can overrun checkerboard regions, and in fact collisions of opposing types of this strong propagation give rise to the (hard uncertainty) blinker boundaries in the first place. The final configuration is the result of the interactions in this distributed computation.

In all of these CAs we expect a framework for local information dynamics to quantitatively confirm these qualitative observations of the computational roles of emergent structure. This will provide a deeper understanding of computation than single or generic measures of bulk statistical behaviour, from which conflict often arises in attempts to provide classification of complex behaviour. In particular, we seek clarification on the long-standing debate regarding the nature of computation in ECA rule 22.

Suggestions that rule 22 is complex include the difficulty in estimating the metric entropy (i.e. temporal entropy rate) for rule 22 in [42], due to "complex long-range effects, similar to a critical phenomenon" [114]. This effectively corresponds to an implication that rule 22 has contains an infinite amount of collective memory (see Sect. 3.1). Also, from an initial condition of only a single "on" cell, rule 22 forms a pattern known as the "Sierpinski Gasket" [92] which exhibits clear self-similar structure. Furthermore, rule 22 is a 1D mapping of the 2D Game of Life CA (known to have the capability for universal computation [104]) and in this sense is referred to as "life in one dimension" [115], and complex structure in the language generated by iterations of rule 22 has been identified [116]. Also, we report here that we have investigated the C_1 complexity measure [117] (an enhanced version of the variance of the input entropy [62]) for all ECAs, and found rule 22 to clearly exhibit the largest value of this measure (0.78 bits to rule 110's 0.085 bits). Similarly, the statistical complexity [20, 51, 65] measure is larger for rule 22 than rules 54 and 110 (at 4.22, 3.80 and 3.91 bits respectively).[14]

On the other hand, suggestions that rule 22 is not complex include its high sensitivity to initial conditions and perturbations leading to Wolfram and Grassberger classifying it as class III chaotic [92, 114]. Gutowitz and Domain [95] claim this renders it as chaotic despite the subtle long-range effects it displays, further identifying its fast statistical convergence, and exponentially long and thin transients in state space (see [62]).

[14] Measured using the CimulA package [100] over 600 time steps of 100000 cells, with light cone depths of 3 time steps.

Importantly, no coherent structures (particles, collisions, etc.) are found for rule 22 using a number of filters (e.g. local statistical complexity [51]). This reflects the *paradigm shift to an examination of local dynamics* rather than generic, overall or averaged analysis. In our approach, we seek to combine this local viewpoint of the dynamics with a quantitative breakdown of the individual elements of computation, and will investigate the computation of rule 22 in this light.

2.3.5 Filtering Structure in Cellular Automata

Measuring the local information dynamics of computation at each space-time point in a CA will create spatiotemporal profiles which can be viewed as a method of filtering the CA. As previously suggested, several methods already exist for filtering the important structural elements (i.e. particles) in CAs [51, 52, 62, 94, 118, 119]. These will provide an important basis for comparison to our spatiotemporal profiles.

The earliest methods were hand-crafted for specific CAs (relying on the user knowing the period of background domains) by Grassberger [118, 119]. Later methods can be automatically applied to any given CA, including finite state transducers to recognise the regular spatial language of the CA using ϵ-machines by Hanson and Crutchfield [94, 120]. Helvik et al. use local information (i.e. local spatial entropy rate) [52]. Wuensche displays executing rules with the most frequently occurring rules filtered out [62]. Shalizi et al. use local statistical complexity (via the light-cone formulation) and local sensitivity [51]. All of these successfully highlight particles against the background domain. Note the use of several *local* information-theoretic measures here (since only local measures, not averages, provide values at every spatiotemporal point).

Obviously, filtering is not a new concept, however the ability to *separately* filter each of the information dynamics of computation would be novel. Most importantly, it would be the first quantitative study of the dynamics of each of these computational operations within the CA. In potentially separating the emergent structures representing each of the information dynamics, this would provide the first quantitative evidence for their computational roles. It would also show how these operations interrelate to give rise to complex behaviour, in comparison to other filters which give only a single view of where that complexity occurs. This would allow a more refined investigation than single measures, and should reveal interesting differences in the parts of the emergent structures that are highlighted. Furthermore, these measures will be generally applicable to any multivariate time-series, unlike some of the filtering measures here (e.g. spatial ϵ-machines [94, 120] and spatial entropy rate [52]) which are only applicable to lattice systems.[15]

[15] Lattice systems are those with a regular spatial ordering for their agents or variables, typically being placed on a one or two-dimensional array.

2.4 The Dynamics of Networks

Complex systems science has been particularly successful in the study of network topology (e.g. see [30, 121]), where the key concepts of small-world [5, 7] and scale-free [6, 122, 123] structures have attracted an enormous amount of attention. In particular, this is because both such topologies are found to be widespread amongst naturally occurring and man-made systems. Small-world topologies balance fixed and random network structures to provide both short path length and high clustering (or from another perspective, provide high global and local efficiency of communication [124]). That small-world networks represent a balance between fixed and random structures resonates with observations that complex systems balance ordered and chaotic properties (see Sect. 2.1.1). Scale-free topologies display a distribution of the degree of nodes (i.e. number of connections to other nodes) that is inversely proportional to the degree (sometimes called a $1/f$ distribution). The arising of scale-free topologies can be explained via the principle of "preferential attachment" [6, 122]—that new nodes introduced to the system preferentially make connections to nodes in proportion to their existing number of connections (because connections to well-connected nodes will prove more rewarding in some sense). A scale-free distribution is highly structured, and is considered to be a signature of self-organised criticality (see [24, 25]).

Yet the time-series behaviour or dynamics on networks have received less attention and are "much less well understood" [9]. There is a deep need for fundamental insights into network dynamics, and how these are related to the underlying structure [125]. Indeed Barabási, who introduced the concept of preferential attachment, states that "we need to tackle the next frontier, which is to understand the dynamics of the processes that take place on networks" [8]. Similarly Watts, who introduced the small-world networks concept, states that "next to the mysteries of dynamics on a network—whether it be epidemics of disease, cascading failure in power systems, or the outbreak of revolutions—the problem of networks that we have encountered up to now are just pebbles on the seashore" [7].

To some degree, the time-series dynamics of state-space trajectories and damage spreading are established, e.g. [126–129]. However, Barabási states that a major problem here is the diversity of types of dynamical process, and wonders whether "these dynamical processes share some common characteristics? I suspect that such commonalities do exist; we just have not yet found the framework to unveil their universality" [8]. He goes on to suggest that if such a framework were to be found, then "combined with the universality of the network topology, we may soon have something that could form the foundation of a theory of complexity" [8].

We believe that the abstract nature of information theory places it as an ideal candidate for investigating and comparing dynamics across different network types, and revealing their commonalities. In particular, the information dynamics of computation align with the way many authors talk about dynamics on networks. This is underlined by Mitchell [9] who suggests that "the main challenge is understanding the dynamics of the propagation of information…in networks, and how these

networks process such information." Later, Mitchell also states that "understanding the ways in which information spreads in networks is one of the most important open problems in science" [2].

These views are echoed in several studies which have investigated the propagation and the processing of information in networks, in particular reporting maximisations of these properties at (approximate) phase transitions between ordered and chaotic regimes. Solé and Valverde [67] investigated the effect of varying the message generation rate in a model of computer networks, finding phase transitions maximising the number of packets actually delivered and the mutual information in the status of random node pairs. They infer that information transfer is maximised at the critical state. Kinouchi and Copelli [80] investigated varying the "branching ratio" (effectively an activity level) in a network of excitable elements, finding phase transitions maximising the dynamic range of the element's output, and inferring a maximisation of information processing at criticality. The spread of information was investigated in [130] under a number assumptions of average diffusion behaviour in a stochastic network model, finding better spread of information in scale-free rather than completely random networks. These investigations align with much conjecture regarding computational properties being maximised at the *edge of chaos* between ordered and chaotic behaviour, e.g. [15, 16, 103].

We are particularly interested in investigating the information dynamics of **random Boolean networks** (RBNs) ([16], and see [131]), in part because of the power in their generality as discrete dynamical network models with a large sample space available. Also, they have a well-known phase transition from ordered to chaotic dynamics, in terms of length of transients in phase space with respect to average connectivity or activity level. We are also motivated by their popularity as models of Gene Regulatory Networks (GRNs). Perhaps most importantly, there have been several recent attempts to study the computational properties of RBNs (in particular information transfer) since they are useful generalised models of computation in networks. Here, Ribeiro et al. [132, 133] measure mutual information in the states of random node pairs as a function of connectivity in the network, and Rämö et al. [61] measure the uncertainty (entropy) in the size of perturbation avalanches as a function of an order parameter. Both find maximisation near the critical point, claiming that their results imply maximisation of information propagation in this regime.

We are also interested in investigating the information dynamics of **cascading failure events** [134, 135]. These are local failures that trigger avalanche mechanisms with large effects over the whole network (e.g. catastrophic blackouts in power grids [134, 136] and global failure in financial markets [125, 137]). Similar to studies of damage spreading or perturbation avalanches on networks (e.g. [129, 138]), investigations of this concept consider how information spreads on the network; indeed: "the phenomena of cascading failures emphasises the need to understand information spreading and how it is affected by network structure" [2]. Often these cascades or avalanches are directly identified with information transfer (e.g. waves of directional change in schooling fish are referred to as "information cascades" [75]). Certainly information transfer is an integral part of cascade dynamics, but the relationship may not be as trivial as a one-to-one mapping. More generally, our interest lies in

examining the way the network *intrinsically* processes information during these extreme events. Make no mistake: during cascading failures, the network is in fact *computing* its new stable state (attractor), so understanding this computation can help understand the dynamics here.

While the aforementioned quantitative studies of computation on networks are interesting, they do not directly measure the information dynamics claimed; e.g. none of the purported measures of information transfer properly measure directed, dynamic flows of information. Measures of model or task specific properties (e.g. in [61, 67, 80, 130]) are qualitatively appealing but give no insights into the underlying quantitative nature of the information dynamics, while mutual information between random pairs of nodes (by Ribeiro et al. [132] and Solé and Valverde [67]) measures dynamic correlation across the collective which may result from an information transfer but is not a measure of it. We also note the more generic measures of "information transfer" and "efficiency in transporting information" presented in [69, 124] respectively, however they are static measures of structure rather than measures of a directed, dynamic flow of information.[16]

A framework for the information dynamics of distributed computation would allow significant insights into computation in networks. In particular, it could be used to produce satisfactory quantitative insights into whether the computational properties of networks are indeed maximised near order-chaos phase transitions. It could also clarify the relationship between damage spreading and information transfer. Furthermore, such a framework would be used to provide quantitative answers on how network structure gives rise to computational properties.

In this section, we introduce RBNs as an important model of time-series dynamics on networks. RBNs are illustrative of complex systems perspectives in this domain. A model for cascading failures on networks will be introduced in Sect. 6.2.1. These two models will be used in Chap. 6 to study the nature of information dynamics in networks and phase transitions.

2.4.1 Random Boolean Networks as a Model of Dynamic Network Behaviour

Random Boolean networks (RBNs) are a class of generic discrete dynamical network models. They are particularly important in artificial life, since they were proposed as models of gene regulatory networks by Kauffman [16]. Indeed Boolean networks have been successfully used to model various GRNs (e.g. the regulatory network controlling metabolism in *E. coli* [139, 140] and the cell-cycle regulatory network of fission yeast [141]). They are also useful as generalised models of computation in networks. See also [131] for another thorough introduction to RBNs.

[16] The work in [69] is part of an interesting trend towards the use of information-theoretic measures to study network topology, e.g. see also [70]. Though information-theoretic, since these measures study topology rather than time-series dynamics they remain out of scope here.

An RBN consists of N nodes in a directed *network* structure. The nodes take *Boolean* state values, and update their state values in time as a deterministic function of the state values of the nodes from which it has incoming links. The network topology (i.e. the adjacency matrix) is determined at *random*, subject to whether the in-degree for each node is constant or stochastically determined given an average in-degree $\overline{K}$ (giving a Poissonian distribution). It is also possible to bias the network structure, e.g. toward scale-free degree distribution [142]. Given the topology, the deterministic Boolean function or lookup table by which each node computes its next state from its neighbours is also decided at *random* for each node, subject to a probability p of producing "1" outputs (p close to 1 or 0 gives low activity, close to 0.5 gives high activity). The nodes here are heterogeneous agents: there is no spatial pattern to the network structure (indeed there is no inherent concept of locality from a global perspective), nor do the nodes have the same update functions.[17] Importantly, the network structure and update functions for each node are held static in time ("quenched"). In classical RBNs (CRBNs), the nodes all update their states synchronously.[18]

It is worth noting that CAs are a sub-class of RBNs, with lattice-style ordering of the nodes and their connections, and homogeneous update rules [127]. So in a similar fashion to CAs, note that the synchronous nature of CRBNs, their Boolean states and deterministic update functions give rise to a global state $\mathbf{X}$ for the network, with deterministic transient trajectories through a state-space ultimately leading to either fixed or periodic attractors in finite-sized networks [127] (and see Sect. 2.3.2). Effectively, the transient is the period in which the network is *computing* its steady state attractor.

RBNs are known to exhibit three distinct phases of dynamics, depending on their parameters: ordered, chaotic and critical. The characteristics of these phases are similar to those described for CAs (see Sect. 2.3.2). At relatively low connectivity (i.e. low degree $\overline{K}$) or activity (i.e. p close to 0 or 1), the network is in an ordered phase, characterised by high stability of states and strong convergence of similar macro states in state space. Alternatively, at relatively high connectivity and activity, the network is in a chaotic phase, characterised by low stability of states and divergence of similar macro states. In the critical phase (described as the *edge of chaos* [15]), there is percolation in nodes remaining static or updating their values, and uncertainty in the convergence or divergence of similar macro states. The phase is also associated with large correlation lengths [103], similar to complex behaviour in CAs.

This phase transition is typically quantified using a measure of sensitivity to initial conditions, or damage spreading. We will describe a measure for this purpose in Sect. 6.1.1. It is important to note that (as per all many-body phase transitions, e.g. in

[17] Though, of course either of these can arise at random.

[18] There has been some debate about the best updating scheme to model GRNs [143], and variations on the synchronous CRBN model are known to produce different behaviours. However, the relevant phase transitions are known to exist in all updating schemes, and their properties depend more on the network size than on the updating scheme [144]. As such, the use of CRBNs is justified for ensemble studies such as ours [128].

the Ising model [145]) RBNs only exhibit a phase transition with truly discontinuous changes in system properties with continuous changes in parameters in the infinite-size limit [132]. Finite-size networks exhibit *approximate* phase transitions with: a continuous change in system properties with respect to parameters [132], variation in these properties over network realisations (which becomes lower at larger network sizes) [61, 132], and a shift of the critical point to a region centred on larger connectivities [144].

Much has been speculated on the possibility that gene regulatory and other biological networks function in (or evolve to) the critical regime (see [131]). It has been suggested that computation occurs more naturally with the balance of order and chaos there [15], possibly with information storage, propagation and processing capabilities maximised [16]. This is of course generalised in the "edge of chaos" hypothesis [15]: that systems exhibiting critical dynamics in the vicinity of a phase transition maximise their computational capability (see earlier discussion in Sect. 2.3.3). Here we seek to improve on previous attempts to measure these computational properties, with a thorough quantitative study of the information dynamics in RBNs.

2.5 Guided Self-Organisation

The principle of self-organisation is well known to offer the advantages of flexibility, robustness and scalability over centralised systems [73]. As such, it is often employed in the design of artificial systems for which these traits are desirable. Most self-organised solutions are currently designed in an ad-hoc manner, since the fundamental nature of self-organisation remains poorly understood. Some designers use an approach specifically tailored to the problem, e.g. [34]. More generally, designers use a genetic algorithm (GA) or programming (GP) approach, with fitness functions measuring specific achievement of the task required of the system (*task-based evolution*), e.g. [146].

Task-based evolution, this incumbent method of designing self-organised systems, can be impractical. Hand-crafting fitness functions for every task can be time-consuming and tedious, and requires specialised human understanding of the task. It has the potential to under-specify the problem (thereby solving a different task) or perhaps over-specify it (leading to an inflexible design). Also, the intelligent designer may not be completely sure of how to measure performance of the required task, or this may be difficult (e.g. measuring speed may require extra sensors). Coupling evolution with learning can be helpful, however specifying rewards for reinforcement learning suffer the same problem regarding measurement of the task [147]. Furthermore, if the initial task-based fitness landscape is flat and features no gradients, task-based evolution has no foothold around which to begin designing a solution. Finally, evolution often delivers intricate solutions of which (human) system managers cannot understand the inner workings: this is particularly undesirable for critical systems where maintenance or prediction of behaviour is required.

As an alternative, the concept of *guided self-organisation* proposes the use of information-theoretic measures to guide the emergence of required information processing structure in self-organised systems [10]. This perspective has been prompted by observations of complexity to grow or necessary information-theoretic structure to emerge during task-based evolution. For example, growth of complexity has been observed during evolution in *artificial life* (ALife) simulation environments, e.g.: by measuring "physical complexity" in Avida [31], and under certain conditions by measuring neural (TSE) complexity of evolved agents in PolyWorld [148–150]. Looking at evolution for particular tasks, Prokopenko et al. [151] observed coordination (measured as excess entropy, see Eq. (2.18)) to increase in snake-like robots evolved for maximum velocity, and [152] observed a decrease in entropy in a swarm evolved for coordinated motion. Also, Ay et al. [153] observe that predictive information (see Eq. (2.20) with $k = 1$) is maximised for an autonomous robot exhibiting behaviour that is both "explorative and sensitive to the environment".

These observations suggest that such information-theoretic measures could be used themselves to guide self-organisation. This idea is fundamentally based on the theory that information structure is vital to the emergence of self-organised intelligence [154]. The concept could provide a consistent framework for the evolutionary design of self-organised systems, using template-based evolution for fundamental computational tasks that underpin the system goal. In theory, this approach would be able to produce useful structure where task-based evolution faces initially flat task-based fitness landscapes, perhaps serving as a platform from which to launch better-equipped task-based evolution. Furthermore, it may provide solutions which are simpler for humans to understand in terms of the underlying information dynamics. Perhaps most important is the potential for this approach to provide insight into the emergence rather than engineering of intelligence [154], and thereby facilitate unsupervised learning.

Several examples of successful guided self-organisation using information-theoretic measures exist in the literature. An interesting example is the concept of *empowerment* presented by Klyubin et al. [155, 156]. This concept refers to an agent's self-perception of its influence over the environment and is measured as the channel capacity of an agent's perception-action loop. Maximisation of empowerment has been shown to induce a necessary structure in an agent's behaviour, and indeed such maximisation has been suggested to be an intrinsic selection pressure.[19] Sporns and Lungarella [157] have evolved hand-eye co-ordination to grab a moving object using maximisation of neural (TSE) complexity. Interestingly, they demonstrated that this solution contained more intrinsic diversity than solutions from task-driven evolution; the increased diversity may afford greater flexibility to the system. Prokopenko et al. [73] were able to evolve fast-moving snake-like robots using maximisation of the excess entropy (see Eq. (2.18)) as an information-theoretic measure of co-ordination.

[19] The justification or otherwise of the suggestion that *natural* evolution is driven by the intrinsic forces of information processing is irrelevant to whether information-driven design can be used as a successful tool for *artificial* systems.

Also, Sperati et al. [158] have observed interesting periodic behaviour and complex structure in groups of robots which were evolved to maximise their mutual information. Note that the "guiding" could be through learning rather than evolution, e.g. the concept of "homeokinesis" [147, 159] which seeks to develop information structure in an agent by having it adapt to minimise the error of an internal model of its behaviour.

We suggest that utilising an understanding of distributed computation is a key approach for guided self-organisation using information-theoretical measures. Fernández and Solé [103] observe that since biological systems perform computations, there is an evolutionary pay-off in nature for this capability. There is good reason to expect this in artificial systems too, since any task we wish the system to achieve involves some form of computation. Indeed, Von Neumann believed that an understanding of distributed computation in CAs "would be essential …for *designing* artificial complex systems" ([88] describing [96]). To be specific, we suggest that the *information dynamics* of distributed computation provide the most intuitive basis here. As previously discussed, these information dynamics are the primitive functions of Turing universal computation, i.e. *information storage*, *transfer* and *modification*. As such, using a framework for distributed computation allows us to *target the design* toward the computational requirements of the task at hand, i.e. selecting either the most relevant computational function as the fitness function, or balancing the functions in a more sophisticated manner. Indeed, this perspective will provide insights to the related field of *unconventional computation*, which specifically seeks to design systems with greater capabilities than traditional computers [160]. Importantly, using such a framework provides a basis through which to understand the computation carried out by the solution. Also, guiding a system toward the building blocks of distributed computation is an intuitive way to facilitate the emergence of collective intelligence.

Information transfer is an important candidate fitness function to be investigated here. It has been conjectured that information transfer can give rise to interesting behaviour and induce necessary structure in a multi-agent system [73]. One inspiration of this viewpoint is the aforementioned concept of empowerment [155, 156], which has been shown to induce necessary structure in an agent's behaviour. This concept is relevant here since it alludes to information transfer in being quantified as the channel capacity between an agent's actuators and sensors through the environment. Also, several authors have attributed a key role to information transfer in facilitating the emergence of complex computation near the critical dynamics of order-chaos phase transitions [15, 17, 67]. Some have inferred maximisation of information transfer in the critical state [17, 67], however evidence has not yet been provided from a directed, dynamic measure of information transfer. Also, information transfer is said to be manifested in "information cascades" spreading across schools of fish [75]. Such mechanisms are biologically crucial, because they can transmit information at a faster speed than that of an incoming predator, with such computational capability providing an evolutionary advantage. We will investigate

the manner in which information transfer can be harnessed in approaches to guided self-organisation in Sect. 8.3.

2.6 Opportunity to Quantify the Information Dynamics of Distributed Computation

In this chapter we have described complex systems, and the growing extent to which the perspective of distributed computation (using measures of information theory) is being used for the analysis and design of these systems. We have focussed on three specific domains (cellular automata, network dynamics and guided self-organisation) in order to describe current understanding of distributed computation. Each of these cases highlighted the specific need for and opportunity to provide a framework to quantify the information dynamics of distributed computation in complex systems. In particular, they have highlighted the need for these operations to be quantified on a local scale in space and time. We hypothesise that if we can describe and quantify distributed computation in terms of information storage, transfer and modification, then we will be better able to understand distributed computation in nature and its sources of complexity.

For CAs in particular, such a framework has the potential to quantitatively confirm the computational role of emergent structures, making a fundamental contribution to our understanding of the nature of distributed computation. As such, CAs will be used as the primary application area to study the measures we propose for this framework in the coming chapters. We emphasise that our approach is not to quantify computation or overall complexity, nor to identify universal computation or determine what is being computed. It is simply intended to quantify the component operations in space-time. We will examine how these operations interact in order to give rise to complex computation.

Similarly, we will investigate whether these information dynamics are maximised in order-chaos phase transitions. Phase transitions in networks provide a particularly important application area for these investigations. Furthermore, this framework has the potential to provide widely-anticipated insights into the dynamics in networks, and how these relate to topology, in a manner that can be compared across network types and address concepts of general interest (e.g. information transfer).

Crucially, considerations of information storage [72–74], transfer [75, 76, 161] and modification [79, 80, 83] extend well beyond these central application areas. We believe that the development of such a framework, and the insights it would provide from these application areas, would have significant impact on the whole of complex systems science. Importantly, it would be relevant in both analysis (e.g. in computational neuroscience as we demonstrate in Sect. 8.2), and design (i.e. in the context of guided self-organisation as we show in Sect. 8.3). In the next chapter we will begin describing our original contribution of this framework by considering the operation of information storage.

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