Chapter 2
Stability of Quasi Two-Dimensional Vortices

J.-M. Chomaz, S. Ortiz, F. Gallaire, and P. Billant

Large-scale coherent vortices are ubiquitous features of geophysical flows. They have been observed as well at the surface of the ocean as a result of meandering of surface currents but also in the deep ocean where, for example, water flowing out of the Mediterranean sea sinks to about 1000 m deep into the Atlantic ocean and forms long-lived vortices named Meddies (Mediterranean eddies). As described by Armi et al. [1], these vortices are shallow (or pancake): they stretch out over several kilometers and are about 100 m deep. Vortices are also commonly observed in the Earth or in other planetary atmospheres. The Jovian red spot has fascinated astronomers since the 17th century and recent pictures from space exploration show that mostly anticyclonic long-lived vortices seem to be the rule rather than the exception. For the pleasure of our eyes, the association of motions induced by the vortices and a yet quite mysterious chemistry exhibits colorful paintings never matched by the smartest laboratory flow visualization (see Fig. 2.1). Besides this decorative role, these vortices are believed to structure the surrounding turbulent flow. In all these cases, the vortices are large scale in the horizontal direction and shallow in the vertical. The underlying dynamics is generally believed to be two-dimensional (2D) in first approximation. Indeed both the planetary rotation and the vertical strong stratification constrain the motion to be horizontal. The motion tends to be uniform in the vertical in the presence of rotation effects but not in the presence of stratification. In some cases the shallowness of the fluid layer also favors the two-dimensionalization of the vortex motion. In the present contribution, we address the following question: Are such coherent structures really 2D? In order to do so, we discuss the stability of such structures to three-dimensional (3D) perturbations paying particular attention to the timescale and the length scale on which they develop. Five instability mechanisms will be discussed, all having received renewed attention in the past few years. The shear instability and the generalized centrifugal instability apply to isolated vortices. Elliptic and hyperbolic instability involve an extra straining effect due to surrounding vortices or to mean shear. The newly discovered zigzag instability
also originates from the straining effect due to surrounding vortices or to mean shear, but is a “displacement mode” involving large horizontal scales yet small vertical scales.

### 2.1 Instabilities of an Isolated Vortex

Let us consider a vertical columnar vortex in a fluid rotating at angular velocity $\Omega$ in the presence of a stable stratification with a Brunt–Väisälä frequency $N^2 = \frac{d\ln \rho}{dz} g$. The vortex is characterized by a distribution of vertical vorticity, $\zeta_{\text{max}}$, which, from now on, only depends on the radial coordinate $r$ and has a maximum value $\eta_{\text{max}}$. The flow is then defined by two nondimensional parameters: the Rossby number $Ro = \frac{\zeta_{\text{max}}}{2 \Omega}$ and the Froude number $F = \frac{\zeta_{\text{max}}}{N}$. The vertical columnar vortex is first assumed to be axisymmetric and isolated from external constrains. Still it may exhibit two types of instability, the shear instability and the generalized centrifugal instability.
2.1.1 The Shear Instability

The vertical vorticity distribution exhibits an extremum:

\[ \frac{d\zeta}{dr} = 0. \]  \hspace{1cm} (2.1)

Rayleigh [44] has shown that the configuration is potentially unstable to the Kelvin–Helmholtz instability. This criterion is similar to the inflexional velocity profile criterion for planar shear flows (Rayleigh [43]). These modes are 2D and therefore insensitive to the background rotation. They affect both cyclones and anticyclones and only depend on the existence of a vorticity maximum or minimum at a certain radius. As demonstrated by Carton and McWilliams [11] and Orlandi and Carnevale [36] the smaller the shear layer thickness, the larger the azimuthal wavenumber \( m \) that is the most unstable. Three-dimensional modes with low axial wavenumber are also destabilized by shear but their growth rate is smaller than in the 2D limit. This instability mechanism has been illustrated by Rabaud et al. [42] and Chomaz et al. [13] (Fig. 2.2).

![Fig. 2.2 Azimuthal Kelvin–Helmholtz instability as observed by Chomaz et al. [13]](image)

2.1.2 The Centrifugal Instability

In another famous paper, Rayleigh [45] also derived a sufficient condition for stability, which was extended by Synge [47] to a necessary condition in the case of axisymmetric disturbances. This instability mechanism is due to the disruption of the balance between the centrifugal force and the radial pressure gradient. Assuming that a ring of fluid of radius \( r_1 \) and velocity \( u_{\theta,1} \) is displaced at radius \( r_2 \) where the velocity equals \( u_{\theta,2} \), (see Fig. 2.3) the angular momentum conservation implies that it will acquire a velocity \( u_{\theta,1}' \) such that \( r_1 u_{\theta,1} = r_2 u_{\theta,1}' \). Since the ambient
pressure gradient at \( r_2 \) exactly balances the centrifugal force associated to a velocity \( u_{\theta,2} \), it amounts to \( \partial p / \partial r = \rho u_{\theta,2}^2 / r_2 \). The resulting force density at \( r = r_2 \) is
\[
\frac{\rho}{r_2} (u'_{\theta,1})^2 - (u_{\theta,2})^2.
\]
Therefore, if \((u'_{\theta,1})^2 < (u_{\theta,2})^2\), the pressure gradient overcomes the angular momentum of the ring which is forced back to its original position, while if on contrary \((u'_{\theta,1})^2 > (u_{\theta,2})^2\), the situation is unstable. Stability is therefore ensured if \( u_{\theta,1}^2 r_1^2 < u_{\theta,2}^2 r_2^2 \). The infinitesimal analog of this reasoning yields the Rayleigh instability criterion
\[
d/dr (u_{\theta} r)^2 \leq 0, \tag{2.2}
\]
or equivalently
\[
\delta = 2 \zeta u_{\theta}/r < 0, \tag{2.3}
\]
where \( \zeta \) indicates the axial vorticity and \( \delta \) is the so-called Rayleigh discriminant. In reality, the fundamental role of the Rayleigh discriminant was further understood through Bayly’s [2] detailed interpretation of the centrifugal instability in the context of so-called shortwave stability theory, initially devoted to elliptic and hyperbolic instabilities (see Sect. 2.3 and Appendix). Bayly [2] considered non-axisymmetric flows, with closed streamlines and outward diminishing circulation. He showed that the negativeness of the Rayleigh discriminant on a whole closed streamline implied the existence of a continuum of strongly localized unstable eigenmodes for which pressure contribution plays no role. In addition, it was shown that the most unstable mode was centered on the radius \( r_{\text{min}} \) where the Rayleigh discriminant reaches its negative minimum \( \delta(r_{\text{min}}) = \delta_{\text{min}} \) and displayed a growth rate equal to \( \sigma = \sqrt{-\delta_{\text{min}}} \).

On the other hand, Kloosterziel and van Heijst [21] generalized the classical Rayleigh criterion (2.3) in a frame rotating at rate \( \Omega \) for circular streamlines. This centrifugal instability occurs when the fluid angular momentum decreases outward:
\[
2r^3 \frac{d}{dr} (r^2 (\Omega + u_{\theta}/r))^2 = (\Omega + u_{\theta}/r) (2\Omega + \zeta) < 0. \tag{2.4}
\]
This happens as soon as the absolute vorticity \( \zeta + 2\Omega \) or the absolute angular velocity \( \Omega + u_{\theta}/r \) changes sign. If vortices with a relative vorticity of a single

---

**Fig. 2.3** Rayleigh centrifugal instability mechanism
sign are considered, centrifugal instability may occur only for anticyclones when the absolute vorticity is negative at the vortex center, i.e., if $Ro^{-1}$ is between $-1$ and $0$. The instability is then localized at the radius where the generalized Rayleigh discriminant reaches its (negative) minimum.

In a rotating frame, Sipp and Jacquin [48] further extended the generalized Rayleigh criterion (2.4) for general closed streamlines by including rotation in the framework of shortwave stability analysis, extending Bayly’s work. A typical example of the distinct cyclone/anticyclone behavior is illustrated in Fig. 2.4 where a counter-rotating vortex pair is created in a rotating tank (Fontane [19]). For this value of the global rotation, the columnar anticyclone on the right is unstable while the cyclone on the left is stable and remains columnar. The deformations of the anticyclone are observed to be axisymmetric rollers with opposite azimuthal vorticity rings.

The influence of stratification on centrifugal instability has been considered to further generalize the Rayleigh criterion (2.4). In the inviscid limit, Billant and Gallaire [9] have shown the absence of influence of stratification on large wavenumbers: a range of vertical wavenumbers extending to infinity are destabilized by the centrifugal instability with a growth rate reaching asymptotically $\sigma = \sqrt{-\delta_{\text{min}}}$.

They also showed that the stratification will re-stabilize small vertical wavenumbers but leave unaffected large vertical wavenumbers. Therefore, in the inviscid stratified case, axisymmetric perturbations with short axial wavelength remain the most unstable, but when viscous effects are, however, also taken into account, the leading

Fig. 2.4 Centrifugal instability in a rotating tank. The columnar vortex on the left is an anticyclone and is centrifugally unstable whereas the columnar vortex on the right is a stable cyclone (Fontane [19])
unstable mode becomes spiral for particular Froude and Reynolds number ranges (Billant et al. [7]).

2.1.3 Competition Between Centrifugal and Shear Instability

Rayleigh’s criterion is valid for axisymmetric modes \((m = 0)\). Recently Billant and Gallaire [9] have extended the Rayleigh criterion to spiral modes with any azimuthal wave number \(m\) and derived a sufficient condition for a free axisymmetric vortex with angular velocity \(u_\theta/r\) to be unstable to a three-dimensional perturbation of azimuthal wavenumber \(m\): the real part of the growth rate

\[
\sigma(r) = -imu_\theta/r + \sqrt{-\delta(r)}
\]

is positive at the complex radius \(r = r_0\) where \(\partial\sigma(r)/\partial r = 0\), where \(\delta(r) = (1/r^3)\partial(r^2u_\theta^2)/\partial r\) is the Rayleigh discriminant. The application of this new criterion to various classes of vortex profiles showed that the growth rate of non-axisymmetric disturbances decreased as \(m\) increased until a cutoff was reached. Considering a family of unstable vortices introduced by Carton and McWilliams [11] of velocity profile \(u_\theta = r \exp(-r^{\alpha})\), Billant and Gallaire [9] showed that the criterion is in excellent agreement with numerical stability analyses. This approach allows one to analyze the competition between the centrifugal instability and the shear instability, as shown in Fig. 2.5, where it is seen that centrifugal instability dominates azimuthal shear instability.

The addition of viscosity is expected to stabilize high vertical wavenumbers, thereby damping the centrifugal instability while keeping almost unaffected

![Fig. 2.5](image_url)

Fig. 2.5 Growth rates of the centrifugal instability for \(k = \infty\) (dashed line) and shear instability for \(k = 0\) (solid line) for the Carton and McWilliams’ vortices [11] for \(\alpha = 4\).
two-dimensional azimuthal shear modes of low azimuthal wavenumber. This may result in shear modes to become the most unstable.

### 2.2 Influence of an Axial Velocity Component

In many geophysical situations, isolated vortices present a strong axial velocity. This is the case for small-scale vortices like tornadoes or dust devils, but also for large-scale vortices for which planetary rotation is important, since the Taylor Proudman theorem imposes that the flow should be independent of the vertical in the bulk of the fluid, but it does not impose the vertical velocity to vanish. In this section, we outline the analysis of [29] and [28] on the modifications brought to centrifugal instability by the presence of an axial component of velocity. As will become clear in the sequel, negative helical modes are favored by this generalized centrifugal instability, when axial velocity is also taken into account.

Consider a vortex with azimuthal velocity component $u_\theta$ and axial flow $u_z$. For any radius $r_0$, the velocity fields may be expanded at leading order:

$$u_\theta(r) = u_\theta^0 + g_\theta(r - r_0), \quad (2.5)$$
$$u_z(r) = u_z^0 + g_z(r - r_0), \quad (2.6)$$

with $g_\theta = \frac{du_\theta}{dr}\bigg|_{r_0}$ and $g_z = \frac{du_z}{dr}\bigg|_{r_0}$. By virtue of Rayleigh’s principle (2.2), axisymmetric centrifugal instability will prevail in absence of axial flow when

$$\frac{g_\theta r_0}{u_\theta^0} < -1, \quad (2.7)$$

thereby leading to the formation of counter-rotating vortex rings.

When a nonuniform axial velocity profile is present, Rayleigh’s argument based on the exchange of rings at different radii should be extended by considering the exchange of spirals at different radii. In that case, these spirals should obey a specific kinematic condition in order for the axial momentum to remain conserved as discussed in [29]. Following his analysis, let us proceed to a change of frame considering a mobile frame of reference at constant but yet arbitrary velocity $\bar{u}$ in the $z$ direction. The flow in this frame of reference is characterized by a velocity field $\bar{u}_z^0$ such that

$$\bar{u}_z^0 = u_z^0 - \bar{u}. \quad (2.8)$$

The choice of $\bar{u}$ is now made in a way that the helical streamlines have a pitch which is independent of $r$ in the vicinity of $r_0$. The condition on $\bar{u}$ is therefore that the distance traveled at velocity $\bar{u}_z^0$ during the time $\frac{2\pi r_0}{u_\theta^0}$ required to complete an entire revolution should be independent of a perturbation $\delta r$ of the radius $r$:
\[
\frac{(\tilde{u}_0^0)(2\pi r_0)}{u_0^0} = \frac{u_z^0 + g_z \delta r (2\pi (r_0 + \delta r))}{u_0^0 + g_\theta \delta r}.
\] (2.9)

Retaining only dominant terms in \(\delta r\), this defines a preferential helical pitch \(\alpha\) for streamlines in \(r_0\) in the co-moving reference frame:

\[
\tan(\alpha) = \frac{\tilde{u}_z^0}{u_0^0} = -\frac{g_z r_0 / u_0^0}{1 - g_\theta r_0 / u_0^0}.
\] (2.10)

In this case, the stream surfaces defined by these streamlines are helical surfaces of identical geometry defining an helical annular tube. This enables [29] to generalize the Rayleigh mechanism by exchanging two spirals in place of rings conserving mass and angular momentum. The underlying geometrical similarity is ensured by the choice of the axial velocity of the co-moving frame. Neglecting the torsion, the obtained flow is therefore similar to the one studied previously. Indeed, the normal to the osculating plane (so-called binormal) is precessing with respect to the \(z\)-axis with constant angle \(\alpha\). Ludwieg [29] then suggests to locally apply the Rayleigh criterion introducing following reduced quantities:

- \(r_0^{\text{eff}} = \frac{r_0}{\cos^2 \alpha}\), the radius of curvature of the helix,
- \(u_\theta^{0,\text{eff}} = \sqrt{(u_\theta^0)^2 + (\tilde{u}_z^0)^2}\), the velocity along streamlines,
- \(g_\theta^{\text{eff}} = g_\theta \cos \alpha + g_z \sin \alpha\), the gradient of effective azimuthal velocity.

Figure 2.6 represents an helical surface inscribed on a cylinder of radius \(r_0\) and circular section \(C'\). The osculating circle \(C\) and osculating plane \(P\) containing the tangent and normal are also shown. The application of the Rayleigh criterion yields

\[
\frac{g_\theta^{\text{eff}} r_0^{\text{eff}}}{u_\theta^{0,\text{eff}}} = \frac{r_0 (g_\theta + g_z \tan \alpha)}{u_0^0} < -1.
\] (2.11)

Using the value of \(\tan \alpha\) (2.10), one is left with

\[
\frac{g_\theta r_0}{u_\theta^0} - \frac{(g_z r_0 / u_0^0)^2}{1 - g_\theta r_0 / u_0^0} < -1,
\] (2.12)

which was found a quarter century after by Leibovich and Stewartson [28], using a completely different and more rigorous method.
Ludwieg [29] thereby anticipated by physical arguments the asymptotic criterion recovered rigorously by Leibovich and Stewartson [28] showing that, when (2.11) holds, the most unstable helices have a pitch:

$$\tan(\alpha) = \frac{k}{m} = -\frac{gzr_0/u_0}{1 - g_\theta r_0/u_0^0}. \quad (2.13)$$

This result was also derived independently in the shortwave asymptotics WKB framework by Eckhoff and Storesletten [18] and Eckhoff [17]. More recently, following the derivation of Bayly [2], LeBlanc and Le Duc [25] have shown how to construct highly localized modes using the WKB description.

### 2.3 Instabilities of a Strained Vortex

In a majority of flows, vortices are never isolated but interact one with each other. They may also interact with a background shear imposed by zonal flow like in the Jovian bands. At leading order, this interaction results in a 2D strain field, $\epsilon$, acting on the vortex and more generally on the vorticity field ($\epsilon$ and $-\epsilon$ are the eigenvalues of the symmetric part of the velocity gradient tensor, the base flow being assumed 2D). The presence of this strain induces two types of small-scale instability.
2.3.1 The Elliptic Instability

Due to the action of the strain field, the vertical columnar vortex is no more axisymmetric but it takes a steady (or quasi-steady) elliptic shape characterized by elliptic streamlines in the vortex core (Fig. 2.7). Following the early works of Moore and Saffman [35], Tsai and Widnall [50], Pierrehumbert [41] Bayly [3], and Waleffe [50] a tremendous number of studies have shown that the strain field induces a so-called elliptic instability that acts at all scales. Readers are referred to the reviews by Cambon [10] and Kerswell [20] for a comprehensive survey of the literature. Here, we shall develop only the local point of view since it gives insights on the instability mechanism and on the effect of stratification and rotation.

For a steady basic flow, with elliptical streamlines, Miyazaki [34] analyzed the influence of a Coriolis force and a stable stratification. The shortwave perturbations are characterized by a wave vector $\mathbf{k}$ and an amplitude vector $\mathbf{a}$. These lagrangian Fourier modes, called also Kelvin waves, satisfy the Euler equations under the Boussinesq approximation (see Appendix for detailed calculation without stratification). Following Lifschitz and Hameiri [33], the flow is unstable if there exists a streamline on which the amplitude $a$ is unbounded at large time.

The system evolving along closed trajectories is periodic, and stability may be tackled by Floquet analysis. In the particular case of small strain, Leblanc [23], following Waleffe [50], gives a physical interpretation of elliptical instability in terms of the parametric excitation of inertial waves in the core of the vortex. The instability problem reduces to a Mathieu equation (2.50) (see Sect. 2.7.5), parametric excitations are found to occur for

$$
\left(\frac{\zeta^2}{4}\right) j^2 = N^2 \sin^2 \theta + (\zeta + 2\Omega)^2 \cos^2 \theta,
$$

where $\theta$ is the angle between the wave vector $\mathbf{k}$ and the spanwise unit vector and $j$ is an integer. Without stratification and rotation, we retrieve for $j = 1$, that, at

![Fig. 2.7 Flow around an elliptic fixed point](image)
small strain, the resonant condition (2.14) is fulfilled only for an angle of $\pi/3$ as demonstrated by Waleffe [50].

For a strain which is not small, the Floquet problem is integrated numerically. Extending Craik’s work [15] Miyazaki [34] observed that the classical subharmonic instability of Pierrehumbert [41] and Bayly [3] ($j = 1$) is suppressed when rotation and stratification effects are added. Other resonances are found to occur. According to the condition (2.14), resonance does not exist when either

$$\frac{\zeta}{2} < \min (N, |\zeta + 2\Omega|) \quad \text{or} \quad \frac{\zeta}{2} > \max (N, |\zeta + 2\Omega|). \quad (2.15)$$

The vortex is then stable with respect to elliptic instability if (Miyazaki [34])

$$F > 2 \quad \text{and} \quad -2 < RO < -2/3 \quad \text{or} \quad F < 2 \quad \text{and} \quad (Ro < -2 \quad \text{or} \quad Ro > -2/3). \quad (2.16)$$

The instability growth rate is (Kerswell [20])

$$\sigma = \frac{\varepsilon (3Ro + 2)^2 (F^2 - 4)}{16 (F^2 (Ro + 1)^2 - 4Ro^2)}. \quad (2.17)$$

The flow is then unstable with respect to hyperbolic instability in the vicinity of $Ro = -2$:

$$\frac{-2}{(1 - 2\varepsilon/\zeta)} < Ro < \frac{-2}{(1 + 2\varepsilon/\zeta)}. \quad (2.18)$$

We want to emphasize that a rotating stratified flow is characterized by two timescales $N^{-1}$ and $\Omega^{-1}$. If we consider the effect of a strain field on a uniform vorticity field, two timescales are added $\varepsilon^{-1}$ and $\zeta^{-1}$ but no length scale. This explains why all the modes are destabilized in a similar manner, no matter how large the wave vector is.

Indeed in the frame rotating with the vortex core (i.e., at an angular velocity $\zeta/2 + \Omega$) the Coriolis force acts as a restoring force and is associated with the propagation of inertial waves. When the fluid is stratified, the buoyancy is a second restoring force and modifies the properties of inertial waves, these two effects combine in the dispersion relation for propagating inertial-gravity waves. The local approach has been compared with the global approach by Le Dizès [26] in the case of small strain and for a Lamb–Oseen vortex.

In the frame rotating with the vortex core, the strain field rotates at the angular speed $-\zeta/2$ and since the elliptic deformation is a mode $m = 2$, the fluid in the core of the vortex “feels” consecutive contractions and dilatations at a pulsation $2\zeta/2$ (i.e., twice faster than the strain field). These periodic constrains may destabilize inertial gravity waves via a subharmonic parametric instability when their pulsations equal half the forcing frequency. If the deformation field were tripolar instead of
dipolar, the resonance frequency would have been $3\zeta/4$ but the physics would have been the same (Le Dizès and Eloy [27], Eloy and Le Dizès [30]). Elliptical instability in an inertial frame occurs for oblique wave vectors and thus needs pressure contribution. When rotation is included, for anticyclonic rotation the most unstable wave vector becomes a purely spanwise mode with $\theta = 0$. In that case, the contribution of pressure is not necessary and disappears from the evolution system. Those modes are called pressureless modes (see Sect. 2.7.4).

The influence of an axial velocity component in the core of a strained vortex was analyzed by Lacaze et al. [22]. They showed that the resonant Kelvin modes $m = 1$ and $m = -1$, which are the most unstable in the absence of axial flow, become damped as the axial flow is increased. This was shown to be due to the appearance of a critical layer which damps one of the resonant Kelvin modes. However, the elliptic instability did not disappear. Other combinations of Kelvin modes $m = -2$ and $m = 0$, then $m = -3$, and $m = -1$ were shown to become progressively unstable for increasing axial flow.

### 2.3.2 The Hyperbolic Instability

The hyperbolic instability is easier to understand for fluid without rotation and stratification. Then, when the strain, $\epsilon$, is larger than the vorticity, $\zeta$, the streamlines are hyperbolic as shown in Fig. 2.8 and the continuous stretching along the unstable manifold of the stagnation point of the flow induces instability. The instability

![Flow around an hyperbolic fixed point](image)
modes have only vertical wave vectors and therefore the modes are “pressureless” since they are associated with zero pressure variations. This instability has been discussed in particular by Pedley [40], Caulfield and Kerswell [12], and Leblanc and Cambon [24]. Like for the previous case, no external length scales enter the problem and the hyperbolic instability affects the wave vectors independently of their modulus and is associated with a unique growth rate $\sigma$ (see Sect. 2.7.2), including background rotation:

$$\sigma^2 = \epsilon^2 - (2\Omega + \zeta/2)^2. \quad (2.19)$$

The stratification plays no role in the hyperbolic instability because the wave vector is vertical and thus the motion is purely horizontal. In the absence of background rotation, the hyperbolic instability develops only at hyperbolic points. In contrast, in the presence of an anticyclonic mean rotation, the hyperbolic instability can develop at elliptical points since $\sigma$ may be real while $\zeta/2$ is larger than $\epsilon$ (see also Sect. 2.7.4).

2.4 The Zigzag Instability

All the previously discussed 3D instability mechanisms, except the 2D Kelvin–Helmholtz instability, are active at all vertical scales and preferentially at very small scales. Their growth rate scales like the inverse of the vortex turnover time. The last instability we would like to discuss has been introduced by Billant and Chomaz [4]. It selects a particular vertical wave number and has been proposed as the basic mechanism for energy transfer in strongly stratified turbulence. Thus we will first discuss the mechanism responsible for the zigzag instability in stratified flows in the absence of rotation. Next, rotation effects will be taken into account.

2.4.1 The Zigzag Instability in Strongly Stratified Flow Without Rotation

When the flow is strongly stratified the buoyancy length scale $L_B = U/N$ is assumed to be much smaller than the horizontal length scale $L$. In that case the vertical deformation of an iso-density surface is at most $L_B^2/L_V$ (where $L_V$ is the vertical scale) and therefore the velocity, which in the absence of diffusion should be tangent to the iso-density surface, is to leading order horizontal.

If we further assume, as did Riley et al. [46] and Lilly [31], that the vertical scale $L_V$ is large compared to $L_B$, then the vertical stretching of the potential vorticity is negligible, since the vertical vorticity itself is (to leading order) conserved while being advected by the 2D horizontal flow. Similarly the variation of height of a column of fluid trapped between two iso-density surfaces separated by a distance
LV is negligible, since the conservation of mass imposes to leading order that the horizontal velocity field is divergence free.

The motion is therefore governed to leading order by the 2D Euler equations independently in each layer of vertical size LV as soon as LV >> LB. To leading order, there is no coupling in the vertical. Having made this remark, Riley et al. [46] and Lilly [31] conjectured that the strongly stratified turbulence should be similar to the purely 2D turbulence and they invoked the inverse energy cascade of 2D turbulence to interpret measured velocity spectra in the atmosphere.

However, Billant and Chomaz [5] have shown that a generic instability is taking the flow away from the assumption LV >> LB. The key idea is that there is no coupling across the vertical if the vertical scale of motion is large compared to the buoyancy length scale. Thus, we may apply to the vortex any small horizontal translations with a distance and possibly a direction that both vary vertically on a large scale compared to LB. This means that, in the limit where the vertical Froude number FV = LB/LV = kLB goes to zero, infinitesimal translations in any directions are neutral modes since they transform a solution of the leading order equation into another solution. Now if FV = kLB is finite but small it is possible to compute the corrections and determine if the neutral mode at kLB = 0 is the starting point of a stable or an unstable branch (see Billant and Chomaz [6]). Such modes are called phase modes since they are reminiscent of a broken continuous invariance (translation, rotation, etc.).

More precisely, in the case of two vortices of opposite sign, a detailed asymptotic analysis leads to two coupled linear evolution equations for the y position of the center of the dipole η(z, t) and the angle of propagation φ(z, t) (see Fig. 2.9) up to fourth order in FV:

\[
\frac{\partial \eta}{\partial t} = \phi,  \\
\frac{\partial \phi}{\partial t} = (D + F_h^2 g_1)F_V^2 \frac{\partial^2 \eta}{\partial z^2} + g_2F_V^4 \frac{\partial^4 \eta}{\partial z^4},
\]

(2.20)  \hspace{2cm} (2.21)

Fig. 2.9 Definition of the phase variables η and φ for the Lamb dipole, from Billant and Chomaz [6]
where $F_h = L_B/L$ is the horizontal Froude number and $D = -3.67, g_1 = -56.4$, and $g_2 = -16.1$. These phase equations show that when $F_V$ is non-zero, the translational invariance in the direction perpendicular to the traveling direction of the dipole (corresponding to the phase variable $\eta$) is coupled to the rotational invariance (corresponding to $\phi$). Substituting perturbations of the form $(\eta, \phi) = (\eta_0, \phi_0) \exp(\sigma t + ikz)$ yields the dispersion relation
\[
\sigma^2 = -(D + g_1 F_h^2) F_V^2 k^2 + g_2 F_V^4 k^4.
\tag{2.22}
\]

Perturbations with a sufficiently long wavelength ($F_V \ll 1$) are always unstable because the coefficients $D$ and $g_1$ are negative. There is, however, a stabilization at large wavenumbers since $g_2$ is negative. Therefore, because the similarity parameter in (2.22) is $k F_V$, the selected wavelength will scale like $L_B$ whereas the growth rate will stay constant and scale like $U/L$. This instability therefore invalidates the initial assumption that the vertical length scale is large compared to the buoyancy length scale. Similar phase equations have been obtained for two co-rotating vortices [39]. In this case, the rotational invariance is coupled to an invariance derived from the existence of a parameter describing the family of basic flows: the separation distance $b$ between the two vortex centers. This leads to two phase equations for the angle of the vortex pair $\alpha(z, t)$ and for $\delta b(z, t)$ the perturbation of the distance separating the two vortices:
\[
\frac{\partial \alpha}{\partial t} = -\frac{2\Gamma}{\pi b^3} \delta b + \frac{\Gamma}{\pi b} D_0 F_V^2 \frac{\partial^2 \delta b}{\partial z^2},
\tag{2.23}
\]
\[
\frac{\partial \delta b}{\partial t} = -\frac{\Gamma b}{\pi} D_0 F_V^2 \frac{\partial^2 \alpha}{\partial z^2},
\tag{2.24}
\]
where $\Gamma$ is the vortex circulation and $D_0 = (7/8) \ln 2 - (9/16) \ln 3$ is a coefficient computed from the asymptotics. The dispersion relation is then
\[
\sigma^2 = -\frac{\Gamma^2}{\pi^2} \left( \frac{2}{b^2} D_0 (F_V k)^2 + D_0^2 (F_V k)^4 \right).
\tag{2.25}
\]
There is a zigzag instability for long wavelengths because $D_0$ is negative. This theoretical dispersion relation is similar to the previous one for a counter-rotating vortex pair except that the most amplified wavenumber depends not only on $F_V$ but also on the separation distance $b$. This is in very good agreement with results from numerical stability analyses [37].

For an axisymmetric columnar vortex, the phase mode corresponds to the azimuthal wave number $m = 1$, and at $k L_B = 0$ the phase mode is associated to a zero frequency. In stratified flows, as soon as a vortex is not isolated, this phase mode may be destabilized by the strain due to other vortices.
2.4.2 The Zigzag Instability in Strongly Stratified Flow with Rotation

If the fluid is rotating, Otheguy et al. [38] have shown that the zigzag instability continues to be active with a growth rate almost constant (Fig. 2.10). However, the wavelength varies with the planetary rotation $\Omega$ and scales like $|\Omega| L/N$ for small Rossby number in agreement with the quasi-geostrophic theory. The zigzag instability then shows that quasi-geostrophic vortices cannot be too tall as previously demonstrated by Dritschel and de la Torre Juárez [16].

\[
\text{Fig. 2.10} \quad \text{Growth rate of the zigzag instability normalized by the strain rate } S = \Gamma/(2\pi b^2) \text{ plotted against the vertical wavenumber } k_z \text{ scaled by the separation distance } b \text{ for } F_h = \Gamma/(2\pi R^2 N) = 0.5 \text{ (} R \text{ is the vortex radius), } Re = \Gamma/(2\pi \nu) = 8000, R/b = 0.15 \text{ and for } Ro = \Gamma/(2\pi R^2 \Omega) = \infty \text{ (+), } Ro = \pm 2.5 \text{ (\triangledown), } Ro = \pm 1.25 \text{ (\circ), } Ro = \pm 0.25 \text{ (\triangle). Cyclonic rotations are represented by filled symbols whereas anticyclonic rotations are represented by open symbols. From [38].}
\]

2.5 Experiment on the Stability of a Columnar Dipole in a Rotating and Stratified Fluid

This last section presents results of an experiment on a vortex pair in a rotating and stratified fluid [14, 8] that illustrates many of the instabilities previously discussed that tends to induce 3D motions.

2.5.1 Experimental Setup

As in Billant and Chomaz [4] a tall vertical dipole is created by closing a double flap apparatus as one would close an open book (Fig. 2.11). This produces a dipole
Fig. 2.11 Sketch of the experimental setup that was installed on the rotating table of the Centre National de Recherches Météorologiques (Toulouse). The flaps are 1 m tall and the tank is 1.4 m long, and 1.4 m large, 1.4 m deep [14, 8]

Control parameters:

\[ F = \frac{U}{L N} \]
\[ F\zeta = \frac{\zeta}{N} \]

\[ R_o = \frac{\zeta}{2\Omega} \]

\[ Re = \frac{UL}{v} \]

Fig. 2.12 Flow parameters for a dipole in a stratified or rotating fluid that moves away from the flaps and, in the absence of instability, is vertical. Particle image velocimetry (PIV) measurements provide the dipole characteristics that are used to compute the various parameters (Fig. 2.12).

### 2.5.2 The State Diagram

Depending upon the value of the Rossby number \( R_o \) and Froude number \( F_{\zeta} \), the different types of instabilities described in the previous sections are observed (Fig. 2.13). Positive Rossby numbers correspond to instabilities observed on cyclonic
vortices, while negative Rossby numbers correspond to instabilities observed on anticyclones. For large Rossby number, Colette et al. [14] and Billant et al. [8] have observed the zigzag instability at small Froude number and the elliptic instability at large Froude number on both vortices as in Billant and Chomaz [4]. As the Rossby number is decreased, the elliptic instability develops with different wavelengths and growth rates on the cyclone and the anticyclone. For smaller Rossby number, the elliptic instability continues to be observed on the cyclone but tends to be stabilized by rotation effects beyond a given Froude number. In contrast, the anticyclone becomes subjected to two other types of instability: a centrifugal instability for large Froude number and an oscillatory asymmetric instability for moderate Froude number.

2.6 Discussion: Instabilities and Turbulence

Experimental results as well as theoretical results show that when the strain field is large enough, a quasi-two-dimensional vortex is never stable versus 3D instabilities. Regarding the elliptic, hyperbolic, and centrifugal instabilities, if the strain is small, only anticyclones are stable in a narrow band between $\max[-1, -(1/2 + \epsilon/\zeta)^{-1}] < Ro < -2/3$ if $F > 2$ and if $F < 2$, vortices are stable for $Ro > -2/3$. All the instabilities which have been described have a growth rate scaling like the vorticity magnitude or strain field induced by the other vortices. This means that these instabilities are as fast as the mechanisms usually invoked for the turbulence cascade,
such as the pairing of same sign vortices or the creation of vorticity filaments. They should therefore modify the phenomenology of the turbulence. In particular for large Rossby number and small Froude number, the pairing of two vortices is unstable to the generalized zigzag instability and, while approaching each other the vortices should form thinner and thinner layers resulting in an energy transfer to smaller length scales and not larger scales as in the 2D turbulence. This effect of the zigzag instability would then explain the result of Lindborg [32] who shows, processing turbulence data collected during airplane flights and computing third-order moment of the turbulence, that the energy cascade in the horizontal energy spectra is direct and not reverse as it was conjectured by Lilly [31].

2.7 Appendix: Local Approach Along Trajectories

In this appendix, we investigate the instability of a 2D steady basic flow characterized by a velocity $U_B$ and a pressure $P_B$ in the inviscid case. The normal vector of the flow field is denoted $e_z$. The 3D perturbations are denoted $(u, p)$. In a frame, rotating at the angular frequency $\Omega = \Omega e_z$, the linearized Euler equation read as

$$\frac{\partial u}{\partial t} + u \cdot \nabla U_B + U_B \nabla u = -\nabla p - 2\Omega \times u,$$  \hspace{1cm} (2.26)

$$\nabla \cdot u = 0.$$  \hspace{1cm} (2.27)

Following Lifschitz and Hameiri [33], we consider a rapidly oscillating localized perturbation:

$$u = \exp \{i \phi(x,t)/\xi\} a(t) + O(\delta),$$ \hspace{1cm} (2.28)

$$p = \exp \{i \phi(x,t)/\xi\} (\pi(t) + O(\delta)),\hspace{1cm} (2.29)$$

where $\xi$ is a small parameter. Substitution in the linearized Euler equations leads to a set of differential equations for the amplitude and the wave vector $k = \nabla \phi$ evolving along the trajectories of the basic flow:

$$\frac{dx}{dt} = U_B,$$  \hspace{1cm} (2.30)

$$\frac{dk}{dt} = -L_B^T k,$$ \hspace{1cm} (2.31)

$$\frac{da}{dt} = \left(\frac{2k k^T}{|k|^2} - I\right) L_B a + \left(\frac{k k^T}{|k|^2} - I\right) C a,$$ \hspace{1cm} (2.32)

with $L_B = \nabla U_B$ the velocity gradient tensor, the superscript $T$ denoting the transposition, $I$ the unity tensor, and $C$ the Coriolis tensor.
\[ C = \begin{pmatrix} 0 & -2\Omega & 0 \\ 2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (2.33)

The pressure has been eliminated by applying the operator:

\[ \mathcal{P}(k) = \left( I - \frac{kk^T}{|k|^2} \right). \] (2.34)

The incompressibility equation (2.27) yields

\[ k . a = 0. \] (2.35)

The stability is analyzed looking for the behavior of the velocity amplitude \( a \). Following Lifschitz and Hameiri [33], the flow is unstable if there exists a trajectory on which the amplitude \( a \) is unbounded at large time.

### 2.7.1 Centrifugal Instability

For flows with closed streamlines, centrifugal instability can be understood by considering spanwise perturbations, as they have been shown to be the most unstable in centrifugal instability studies (see [2] or [48] for a detailed discussion). If \( k(t = 0) = k_0 e_z \), (2.31) yields \( k(t) = k_0 e_z \); the base flow, evolving in a plane perpendicular to \( e_z \), does not impose tilting or stretching along \( z \) direction. The incompressibility equation (2.35) gives \( a_z = 0 \).

In the plane of the steady base flow, the trajectories (streamlines in the steady case) may be referred to as streamline function value \( \psi \). The two vectors, \( U_B \) and \( \nabla\psi \), provide an orthogonal basis. The amplitude equation in the plane perpendicular to \( e_z \) may be expressed in this new coordinate system, and following [48], (2.32) becomes

\[
\frac{d}{dt} \begin{pmatrix} a . U_B \\ a . \nabla\psi \end{pmatrix} = \begin{pmatrix} 0 & \zeta + 2\Omega \\ -2(V \frac{\nabla}{\nabla} + \Omega) & 2V'/V \end{pmatrix} \begin{pmatrix} a . U_B \\ a . \nabla\psi \end{pmatrix},
\] (2.36)

with \( V = |U_B(x)| \), \( V' \) the lagrangian derivative \( \frac{d}{dt} V = U_B . \nabla V \) and \( \mathcal{R} \) the local algebraic curvature radius defined by [48]

\[
\mathcal{R}(x, y) = \frac{V^3}{\nabla\psi . (U_B . \nabla U_B)}. \] (2.37)

Note that the generalized Rayleigh discriminant, \( \delta = 2 \left( \frac{V}{\mathcal{R}} + \Omega \right) (\zeta + 2\Omega) \), appears in (2.36) as the opposite of the determinant of the governing matrix. Sipp and Jacquin [48] showed that if there exists a particular streamline on which \( \delta < 0 \), the velocity amplitude, \( a \), is unbounded at large time. However, the general proof for
the existence of diverging solutions of (2.36) for a closed (non-circular) streamline with $\delta$ negative requires further mathematical analyses.

### 2.7.2 Hyperbolic Instability

In this section, we focus again on a spanwise perturbation characterized by a wavenumber $k$ perpendicular to the flow field. The contribution of the pressure disappears in that case and the instability is called pressureless. Equation (2.32) reduces to

$$\frac{da}{dt} = - L_B a - C a. \tag{2.38}$$

We look for instability of a basic flow characterized by

$$L_B = \begin{pmatrix} 0 & \epsilon - \xi \frac{1}{2} & 0 \\ \epsilon + \xi \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.39}$$

with $\epsilon$ the local strain and $\zeta$ the vorticity of the base flow. For $|\epsilon|$ greater than $|\zeta|/2$ the streamlines are hyperbolic. The integration of (2.38) is straightforward. The perturbed velocity amplitude $a$ has an exponential behavior, $\exp(\sigma t)$, with $\sigma$

$$\sigma = \sqrt{\epsilon^2 - (\zeta/2 + 2\Omega)^2}, \tag{2.40}$$

as discussed in Sect. 2.3.2.

### 2.7.3 Elliptic Instability

The basic flow is defined by the constant gradient velocity tensor (2.39) but with $|\zeta|/2$ greater than $|\epsilon|$. Trajectories are found to be elliptical with an aspect ratio or eccentricity:

$$E = \sqrt{\frac{\xi \frac{1}{2} - \epsilon}{\sqrt{\xi \frac{1}{2} + \epsilon}}} \tag{2.41}$$

Along those closed trajectories, (2.31) has period

$$T = \frac{2\pi}{Q} = \frac{2\pi}{\sqrt{(\zeta/2)^2 - \epsilon^2}} \tag{2.42}$$
The wave vector solution of (2.31),

\[ k = k_0 \left( \sin(\theta) \cos(Q(t - t_0)), \ E \ sin(\theta) \sin(Q(t - t_0)), \ \cos(\theta) \right),\] (2.43)

with \( k_0 \) and \( t_0 \) integration constants, is easily obtained (see, for example, Bayly [3]). The wave vector describes an ellipse parallel to the \((x, y)\) plane, with the same eccentricity as the trajectories ellipse, but with the major and minor axes reversed. The stability is investigated with Floquet theory, looking for the eigenvalues of the monodromy matrix \( \mathcal{M}(T) \), which is a solution of

\[
\frac{d\mathcal{M}}{dt} = \left( \frac{2kk^T}{|k|^2} - \mathcal{I} \right) L_B \mathcal{M} + \left( k k^T \frac{|k|^2}{|k|^2} - \mathcal{I} \right) C \mathcal{M}, \] (2.44)

with \( \mathcal{M}(0) = \mathcal{I}, \) (2.45)

integrated over one period \( T \). The basic flow lies in the \((x, y)\) plane, so that one of the eigenvalues of the monodromy matrix is \( m_3 = 1 \). The average of the trace of the matrix \( \left( \frac{2kk^T}{|k|^2} - \mathcal{I} \right) \mathcal{L}_B + \left( k k^T \frac{|k|^2}{|k|^2} - \mathcal{I} \right) \mathcal{C} \) is zero, making the determinant of \( \mathcal{M}(T) \) equal to unity. The two other eigenvalues of \( \mathcal{M}(T) \) are then \( m_1 m_2 = 1 \), with \( m_{1,2} \) either complex conjugate indicating stable flow, or real and inverse for unstable flow. Generally, \( m_{1,2} \) are obtained by numerical integration of (2.44) and (2.45). Note, however, that the following two cases may be tackled analytically.

### 2.7.4 Pressureless Instability

In the case of a pressureless instability the monodromy equation is a solution of

\[
\frac{d\mathcal{M}}{dt} = - \mathcal{L}_B \mathcal{M} - \mathcal{C} \mathcal{M}, \] (2.46)

\[ \mathcal{M}(0) = \mathcal{I}. \] (2.47)

The eigenvalues of the monodromy equation are

\[
m_i = \exp \left( \pm \sqrt{\epsilon^2 - (\xi/2 + 2\Omega)^2 T} \right), \quad \text{with } i = 1, 2. \] (2.48)

With rotation, the elliptical flow may be unstable to pressureless instabilities, as discussed in Sects. 2.3.1 and 2.3.2 and in Craik [15].

### 2.7.5 Small Strain \( |\epsilon| << 1 |\)

Following Waleffe [50], we derive from (2.32) the equation for the rescaled component of the velocity along the \( z \)-axis denoted \( q \):
\[ q = a_3 \frac{|k|^2}{|k//|^2}, \quad (2.49) \]

with \( k// \) the component of the wavenumber on the \((x, y)\) plane. For small strain, this equation leads to a Mathieu equation with a rescaled time, \( t^* = \zeta t \),

\[ \frac{d^2 q}{dt^{*2}} + (\alpha + 2\epsilon b \sin t^*) q = 0, \quad (2.50) \]

where

\[ \alpha = \left( \frac{Ro + 1}{Ro} \right)^2 \cos^2 \theta, \quad (2.51) \]

\( Ro = \frac{\zeta}{\Omega_2} \) is the Rossby number and

\[ b = \frac{1}{2} \left[ \left( \frac{Ro + 1}{Ro} \right)^2 \sin^2 \theta + \frac{1}{Ro} + \frac{3}{2} \right] \cos^2 \theta. \quad (2.52) \]

Parametric resonances occur when

\[ \alpha = \frac{1}{4} j^2, \quad (2.53) \]

with \( j \) an integer, giving the formula (2.14) discussed in Sect. 2.3.1.

**References**

44. Rayleigh, L.: On the instability of cylindrical fluid surfaces. Phil. Mag. 34, 177 (1892). 37